Math 323: Practice for Midterm 4

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §23 and §28-29, §31-34. Knowledge of earlier material will also be necessary to do well on the test, but earlier material will not be explicitly tested. You are expected to know all material covered in the course up until now.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

power series, radius of convergence, interval of convergence, converges pointwise, differentiable at a, differentiable, derivative, strictly increasing, increasing, strictly decreasing, decreasing, partition, upper and lower Darboux sums, upper and lower Darboux integrals, (Darboux) integrable, Darboux integral

Theorems. Theorems given names in the book are often the most important. Theorems (and similar results) you may be required to state:

18.2 Intermediate Value Theorem, 28.4 Chain Rule, 29.2 Rolles Theorem, 29.3 Mean Value Theorem, 29.8 Intermediate Value Theorem for Derivatives, 31.3 Taylors Theorem, 33.1 Monotonic functions are integrable, 33.2 Continuous functions are integrable, 33.9 Intermediate Value Theorem for Integrals, 34.1 Fundamental Theorem of Calculus I, 34.2 Integration by Parts, 34.3 Fundamental Theorem of Calculus II.

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test.

1. (a) Complete the following definition:

   A real valued function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if...

   Solution: $x < y$ implies $f(x) < f(y)$.

   (b) State the Mean Value Theorem.

   Solution: Let f be a real-valued function which is continuous on the closed interval $[a, b]$ and differentiable on (a, b). Then, there is a $c \in (a, b)$ so that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   (c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is strictly increasing.

Solution: Here are two solutions:
Direct proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$. Fix arbitrary $a, b \in \mathbb{R}$ with $a < b$. We will show that $f(a) < f(b)$. By the Mean value theorem, there is a $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

By hypothesis, $f'(c) > 0$. Also, we know that $b - a > 0$. Therefore, we must have $f(b) - f(a) > 0$ and $f(a) < f(b)$.
Proof by contradiction. Suppose to the contrary that f is not strictly increasing. Then, there are real numbers a and b with $a < b$ and $f(a) \geq f(b)$. Then by the Mean Value Theorem, there is a $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

But this contradicts the assumption $f'(x) > 0$ for all x.

2. Consider the following definitions:

Let f be a real-valued function defined on an open interval containing $a \in \mathbb{R}$. The function is *rightward increasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \geq f(a)$. Similarly, the function is *rightward decreasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \leq f(a)$.

(a) Prove that $f(x) = x - x^3$ is rightward increasing at $a = 0$.

Solution: Let $\delta = 1$. We will show that $0 < x < 1$ implies that $f(x) \geq f(a) = 0$. Fix x with $0 < x < 1$. By multiplying by $x > 0$, we know that $0 < x^2 < x < 1$. Therefore by negation, we have $-1 < -x^2 < 0$. Then by adding one, $0 < 1 - x^2 < 1$. Now we multiply through by x again to obtain:

$$0 < f(x) = x - x^3 < x < 1.$$

In particular, $f(x) > 0$ as desired.

(b) Prove that the following function is neither rightward increasing nor rightward decreasing at $a = 0$:

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Solution: We will prove this by contradiction. Assume that the statement is false. Then f is either rightward increasing or rightward decreasing. We will consider each case separately.
Suppose f is rightward increasing. Then, there is a $\delta > 0$ so that $0 < x < \delta$ implies

that $f(x) \geq f(a) = 0$. Fix this δ. Let $n \geq 1$ be an integer so that

$$n > \frac{\frac{1}{\delta} + \frac{\pi}{2}}{2\pi}.$$

Let $x = \frac{1}{2\pi n - \frac{\pi}{2}}$. Because of our choice of n, we have that $x > 0$ and

$$\frac{1}{x} = 2\pi n - \frac{\pi}{2} > 2\pi\left(\frac{\frac{1}{\delta} + \frac{\pi}{2}}{2\pi}\right) - \frac{\pi}{2} = \frac{1}{\delta}.$$

Therefore, $0 < x < \delta$ and by assumption it must be that $f(x) \geq 0$. But,

$$f(x) = x\sin(2\pi n - \frac{\pi}{2}) = -x < 0,$$

which contradicts this assumption.

Now suppose that f is rightward decreasing. Then, there is a $\delta > 0$ so that $0 < x < \delta$ implies that $f(x) \leq f(a) = 0$. Fix this δ. Let $n \geq 0$ be an integer so that

$$n > \frac{\frac{1}{\delta} - \frac{\pi}{2}}{2\pi}.$$

Let $x = \frac{1}{2\pi n + \frac{\pi}{2}}$.Because of our choice of n, we have that $x > 0$ and

$$\frac{1}{x} = 2\pi n + \frac{\pi}{2} > 2\pi\left(\frac{\frac{1}{\delta} - \frac{\pi}{2}}{2\pi}\right) + \frac{\pi}{2} = \frac{1}{\delta}.$$

Therefore, $0 < x < \delta$ and by assumption it must be that $f(x) \leq 0$. But,

$$f(x) = x\sin(2\pi n + \frac{\pi}{2}) = x > 0,$$

which contradicts this assumption.

3. Use the Mean Value Theorem to prove that if $f'(x) = 0$ for all x in the interval $(0, 1)$, then $f(x)$ is constant on $(0, 1)$.

Solution: Suppose f is not constant on $(0, 1)$. Then there are points $x < y$ in $(0, 1)$ with $f(x) \neq f(y)$. Then the Mean value theorem states that there is a $z \in (x, y)$ so that

$$f'(z) = \frac{f(y) - f(x)}{y - x} \neq 0.$$

But this is a contradiction, since we were told that $f'(z) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose f is differentiable at zero with $f'(0) > 0$. Show that there is an $\epsilon > 0$ so that whenever $0 < x < \epsilon$, we have $f(x) > f(0)$.

Solution: Observe that $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x}$. Set $\epsilon = f'(0)$. There is a δ so that $|x| < \delta$ and $x \neq 0$ implies that $|\frac{f(x)-f(0)}{x} - f'(0)| < \epsilon = f'(0)$. Therefore, for any x with $|x| < \delta$ and $x \neq 0$ we have $\frac{f(x)-f(0)}{x} > 0$. Therefore, $f(x) > f(0)$ when $0 < x < \delta$.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Also assume that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty.$$

   (a) Prove that there is an x so that $f'(x) = 0$.
   (b) Prove that there is an x so that $f'(x) > 0$.
   (c) Prove that there is an x so that $f'(x) < 0$.
   (d) Given an example of a function f satisfying the statements above for which $-1 < f'(x) < 1$ for all $x \in \mathbb{R}$.

Solution: Since f is differentiable, it is also continuous. Since $\lim_{x\to+\infty} f(x) = +\infty$, there is an $a > 0$ so that $f(a) > f(0)$. Since $\lim_{x\to-\infty} f(x) = +\infty$, there is a $b < 0$ so that $f(b) > f(0)$.

(b): By the Mean Value theorem, there is an $x \in (0, a)$ with $f'(x) = \frac{f(a)-f(0)}{a} > 0$.

(c): By the Mean Value theorem, there is an $x \in (b, 0)$ with $f'(x) = \frac{f(0)-f(b)}{-b} < 0$.

(a): Let $M = \min\{f(a), f(b)\}$. By the intermediate value theorem, there is an $a' \in [0, a]$ with $f(a') = M$. Similarly, there is a $b' \in [b, 0]$ with $f(b') = M$. Then, by Rolle's theorem applied to the interval $[b', a']$, there is an $x \in (b', a')$ with $f'(x) = 0$.

(d): $f(x) = \sqrt{1 + x^2}$.

6. Let f be a differentiable function defined on $[\frac{1}{4}, \frac{3}{4}]$ so that $f(\frac{1}{4}) = f(\frac{3}{4})$. For each $a \in \mathbb{R}$, define $g_a(x) = ax(1-x)$. We will prove that the graph of f is somewhere tangent to the graph of g_a for some a. (That is, there is an a and an $x \in (\frac{1}{4}, \frac{3}{4})$ so that $f(x) = g_a(x)$ and $f'(x) = g_a'(x)$.)

   (a) Define $h(x) = \frac{f(x)}{x(1-x)}$. Observe that $f(x) = g_a(x)$ whenever $a = h(x)$.
   (b) Show that there is an $x_0 \in (\frac{1}{4}, \frac{3}{4})$ with $h'(x_0) = 0$.
   (c) With x_0 as in part (b), show that there is an a so that $f(x_0) = g_a(x_0)$ and $f'(x_0) = g_a'(x_0)$.

Solution: (a): Suppose $f(x) = g_a(x)$ and $x \in [\frac{1}{4}, \frac{3}{4}]$. Then $f(x) = ax(1-x)$. Therefore, $a = \frac{f(x)}{x(1-x)} = h(x)$.

(b): Observe that $h(\frac{1}{4}) = \frac{16}{3} f(\frac{1}{4})$ and $h(\frac{3}{4}) = \frac{16}{3} f(\frac{3}{4})$. So, since $f(\frac{1}{4}) = f(\frac{3}{4})$, we have $h(\frac{1}{4}) = h(\frac{3}{4})$. Now applying Rolle's theorem to this interval, there is an $x_0 \in (\frac{1}{4}, \frac{3}{4})$ with $h'(x_0) = 0$.

(c): We know that $h'(x_0) = 0$. Set $a = h(x_0)$. Then by part (a), we have $f(x_0) = g_a(x_0)$. We compute $h'(x) = \frac{f'(x)x(1-x)-f(x)(1-2x)}{x^2(1-x)^2}$. Since $h'(x_0) = 0$, we have

$$(*) \qquad f'(x_0)x_0(1-x_0) - f(x_0)(1-2x_0) = 0.$$

Recall $g_a(x_0) = ax_0(1-x_0)$ and $g_a'(x_0) = a(1-2x_0)$. So, we can rewrite $(*)$ as

$$f'(x_0)g_a(x_0) - f(x_0)g_a'(x_0) = 0.$$

Dividing through by $f(x_0) = g_a(x_0)$, we see $f'(x_0) = g_a'(x_0)$, whenever $f(x_0) \neq 0$. When $f(x_0) = 0$, we see that $a = 0$. In this case $(*)$ gives $f'(x_0)x_0(1-x_0) = 0$, so that $f'(x_0) = 0$. Since g_0 is a constant zero function, $g_0'(x_0) = 0$ as well.

7. Let $f(x) = x\sqrt{|x|}$. (This is the product of x and the square root of the absolute value of x.) At which points $x \in \mathbb{R}$ is f differentiable? Prove your answer is correct, and rigorously compute the derivative at all points at which f is differentiable.

Solution: Recall that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Then

$$\text{if } x \geq 0, \text{ then } f(x) = x^{\frac{3}{2}}.$$

Otherwise

$$\text{if } x < 0, \text{ then } f(x) = x\sqrt{-x} = -(\sqrt{-x})^2\sqrt{-x} = -(-x)^{\frac{3}{2}}.$$

Now we will compute derivatives. For $x > 0$, we have

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}.$$

For $x < 0$, we have

$$f'(x) = -\frac{3}{2}(-x)^{\frac{1}{2}}(-1) = \frac{3}{2}\sqrt{-x}.$$

For $x = 0$, we have to consider both cases. By definition f is differentiable at zero if the following limit is a real number:

$$\lim_{t\to 0}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0}\frac{t\sqrt{|t|}}{t} = \lim_{t\to 0}\sqrt{|t|} = 0,$$

with the last step following from the observation that $\sqrt{|x|}$ is continuous at zero. (The function $x \mapsto \sqrt{|x|}$ is continuous because it is the composition of continuous functions.)

The above discussion shows that f is differentiable on all of $\mathbb{R}$. Moreover, we can see by checking cases that the derivative is $f'(x) = \frac{3}{2}\sqrt{|x|}$.

8. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if for any $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ implies

$f(x_1) < f(x_2)$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all x, then f is strictly increasing.

Solution: Assume $f'(x) \geq 0$ for all $x \in \mathbb{R}$. Let $x_1 < x_2$. We will verify that $f(x_1) < f(x_2)$.

By the Mean Value Theorem, there is a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By rearranging terms, we see that

$$f(x_2) - f(x_1) = f'(x)\big(x_2 - x_1\big).$$

Then we see that $f(x_2) - f(x_1)$ is positive because both $f'(x) > 0$ and $x_2 > x_1$. So, we have shown $f(x_2) > f(x_1)$. This proves that f is strictly increasing.

9. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Complete the following definitions:
   (a) For any subset $S \subset [a, b]$,

   $M(f, S) =$

   $m(f, S) =$

   Solution: Look these up in the book.

   (b) A *partition* P of $[a, b]$ is ...
   (c) The *upper and lower Darboux sums* of f with respect the partition P are ...
   (d) The *upper and lower Darboux integrals* of f are ...
   (e) The function f is *integrable* on $[a, b]$ if ...
   (f) If f is integrable on $[a, b]$, the *value of the integral* of f over $[a, b]$ is ...

10. In class lectures and in the book, the following theorem is discussed:

    Theorem. Every monotonic function f on $[a, b]$ is integrable.

    Give a proof of this theorem in the case when f is increasing. You may use basic results about the Darboux integral which were established in class and in the book.

    Solution: See the proof of Theorem 33.1 in the book.

11. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq \frac{1}{2}$ and $f(\frac{1}{2}) = 5$. Use the definition of the Darboux integral to show that f is integrable and compute its integral.

Solution: We will construct a sequence of partitions P_n so that $U(f, P_n)$ and $L(f, P_n)$ converge to the same value. It follows that that value will be the integral.

For an integer $n > 2$ define $P_n = \{0, \frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}, 1\}$. Observe that for any such n:

$$M(f, [0, \frac{1}{2} - \frac{1}{n}]) = m(f, [0, \frac{1}{2} - \frac{1}{n}]) = 0.$$

$$M(f, [0, \frac{1}{2} + \frac{1}{n}]) = m(f, [0, \frac{1}{2} + \frac{1}{n}]) = 0.$$

$$M(f, [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]) = \frac{1}{2} \quad \text{and} \quad m(f, [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]) = 0.$$

Thus, $L(f, P_n) = 0$ for all n and $U(f, P_n) = \frac{2}{n} \cdot 12 = \frac{1}{n}$. Observe that

$$\lim L(f, P_n) = 0 \quad \text{and} \quad \lim U(f, P_n) = 0,$$

so f is integrable on $[0, 1]$ and $\int_0^1 f = 0$.

12. Assume that $f : [0, 1] \to \mathbb{R}$ is a continuous function. Prove that f is integrable.

Solution: See the proof of Theorem 33.2 in the book.

13. Assume that f and g are integrable functions on $[a, b]$ and that $f(x) \geq g(x)$ for all $x \in [a, b]$. Show that $\int_a^b f \geq \int_a^b g$.

Solution: See the proof of Theorem 33.4 in the book.

14. Let f be a real-valued function which is differentiable on $[a, b]$. Let P be a partition of $[a, b]$. Prove that the Darboux sums of the derivative f' satisfy the inequality

$$L(f', P) \leq f(b) - f(a) \leq U(f', P).$$

Solution: Let $P = \{a = t_0 < t_1 < \ldots < t_N = b\}$ be a partition of $[a, b]$. Consider an interval $[t_k, t_{k+1}]$. By the Mean value theorem, there is an $x_k \in (t_k, t_{k+1})$ so that

$$f'(x_k) = \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k}. \tag{1}$$

$$m(f', [t_k, t_{k+1}]) \leq f'(x_k) \leq M(f', [t_k, t_{k+1}]).$$

We will use this to obtain our bounds. Observe:

$$\begin{aligned} U(f', P) &= \textstyle\sum_{k=0}^{N-1} M(f', [t_k, t_{k+1}])(t_{k+1} - t_k) \geq \sum_{k=0}^{N-1} f'(x_k)(t_{k+1} - t_k) \\ &= \textstyle\sum_{k=0}^{N-1} \big(f(t_{k+1}) - f(t_k)\big) = f(b) - f(a), \end{aligned}$$

where we have used (1) and most terms cancel. Similarly,

$$\begin{aligned} L(f',P) &= \textstyle\sum_{k=0}^{N-1} m(f',[t_k,t_{k+1}])(t_{k+1}-t_k) \leq \sum_{k=0}^{N-1} f'(x_k)(t_{k+1}-t_k) \\ &= \textstyle\sum_{k=0}^{N-1} \big(f(t_{k+1})-f(t_k)\big) = f(b)-f(a). \end{aligned}$$

15. Let $f(x) = \cos(x)$.

(a) State a version of Taylor's theorem.

Solution: Theorem 31.3 Taylor's Theorem:
Let f be defined on (a,b) where $a < c < b$; here we allow $a = -\infty$ or $b = +\infty$. Suppose the nth derivative $f^{(n)}$ exists on (a,b). Then for each $x \neq c$ in (a,b) there is some y between c and x such that

$$R_n(x) = f^{(n)}(y)\frac{(x-c)^n}{n!}.$$

(Here $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$.)

(b) Compute the Taylor series for f about zero.

Solution: Observe that

$$f^{(k)}(x) = \begin{cases} \cos x & \text{if } k \equiv 0 \pmod 4 \\ -\sin x & \text{if } k \equiv 1 \pmod 4 \\ -\cos x & \text{if } k \equiv 2 \pmod 4 \\ \sin x & \text{if } k \equiv 3 \pmod 4 \end{cases} \tag{2}$$

So

$$f^{(k)}(0) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 4 \\ 0 & \text{if } k \equiv 1 \pmod 4 \text{ or } k \equiv 3 \pmod 4 \\ -1 & \text{if } k \equiv 2 \pmod 4 \end{cases}$$

Since the expression is only non-zero when k is even, we can let $k = 2j$ and the Taylor series is

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}.$$

(c) Use Taylor's theorem to show that the Taylor series for f converges to f pointwise on $\mathbb{R}$. (You must use Taylor's theorem directly.)

Solution: Fix a "point" $x \in \mathbb{R}$. By Taylor's theorem,

$$R_n(x) = f^{(n)}(y)\frac{x^n}{n!}$$

for some y between 0 and x. Observe that from our derivative formulas (2),

$$|f^{(n)}(y)| \leq 1$$

no matter what y is. Thus,

$$|R_n(x)| \leq \frac{x^n}{n!} \quad \text{for all } n.$$

Observe that no matter what x is we have $\lim \frac{x^n}{n!} = 0$, thus $R_n(x) \to 0$ as $n \to \infty$. Thus $\cos(x)$ equals its Taylor series everywhere.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Suppose that $\lim_{x\to 1^+} f(x) = 7$ and $\lim_{x\to 1^-} f(x) = 3$. Define $F(x) = \int_0^x f(t)\ dt$.

(a) Suppose $y > 1$. Write $F(y) - F(1)$ as an integral of f.

Solution:

$$F(y) - F(1) = \int_0^y f(t)\ dt - \int_0^1 f(t)\ dt = \int_1^y f(t)\ dt.$$

(b) Show that there is a constant $c > 0$ so that $F(y) - F(1) \geq 6(y-1)$ whenever $1 < y < 1+c$.

Solution: This follows from the statement that $\lim_{x\to 1^+} f(x) = 7$. This means that for all $\epsilon > 0$ there is a $\delta > 0$ so that $1 < x < 1+\delta$ implies $|f(x) - 7| < \epsilon$. Taking $\epsilon = 1$ we see that there is a $\delta > 0$ so that $1 < x < 1+\delta$ implies $6 < f(x) < 8$. Then setting $c = \delta$ and taking y so that $1 < y < 1+c$ we have $f(t) \geq 6$ when $1 < t \leq 1+y$ so

$$F(y) - F(1) = \int_1^y f(t)\ dt \geq \int_1^y 6\ dt = 6(y-1).$$

(c) Similarly, it follows that there is a $c > 0$ so that $F(1) - F(y) < 4(1-y)$ whenever $1 - c < y < 1$. (You do not need to prove this.) Use these two facts to show that F is not differentiable at $x = 1$.

Solution: Assume to the contrary that F is differentiable at 1. Then by definition

$$F'(1) = \lim_{y\to 1} \frac{F(y) - F(1)}{y-1}.$$

But from part (b) we have

$$\lim_{y\to 1^+} \frac{F(y) - F(1)}{y-1} \geq 6.$$

From the statement in (c) we have

$$\lim_{y\to 1^-} \frac{F(y) - F(1)}{y-1} \leq 4.$$

This is a contradiction since the limits from the right and the left can not be equal.