Math 323: Practice for Midterm 4

Prof. Hooper

Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §23 and §28-29, §31-34. Knowledge of earlier material will also be necessary to do well on the test, but earlier material will not be explicitly tested. You are expected to know all material covered in the course up until now.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

power series, radius of convergence, interval of convergence, converges pointwise, differentiable at a, differentiable, derivative, strictly increasing, increasing, strictly decreasing, decreasing, partition, upper and lower Darboux sums, upper and lower Darboux integrals, (Darboux) integrable, Darboux integral

Theorems. Theorems given names in the book are often the most important. Theorems (and similar results) you may be required to state:

18.2 Intermediate Value Theorem, 28.4 Chain Rule, 29.2 Rolles Theorem, 29.3 Mean Value Theorem, 29.8 Intermediate Value Theorem for Derivatives, 31.3 Taylors Theorem, 33.1 Monotonic functions are integrable, 33.2 Continuous functions are integrable, 33.9 Intermediate Value Theorem for Integrals, 34.1 Fundamental Theorem of Calculus I, 34.2 Integration by Parts, 34.3 Fundamental Theorem of Calculus II.

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test.

1. (a) Complete the following definition:

   A real valued function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if…

   (b) State the Mean Value Theorem.

   (c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is strictly increasing.

2. Consider the following definitions:

   Let f be a real-valued function defined on an open interval containing $a \in \mathbb{R}$. The function is *rightward increasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \geq f(a)$. Similarly, the function is *rightward decreasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \leq f(a)$.

   (a) Prove that $f(x) = x - x^3$ is rightward increasing at $a = 0$.

   (b) Prove that the following function is neither rightward increasing nor rightward decreasing at $a = 0$:

   $$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

3. Use the Mean Value Theorem to prove that if $f'(x) = 0$ for all x in the interval $(0, 1)$, then $f(x)$ is constant on $(0, 1)$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose f is differentiable at zero with $f'(0) > 0$. Show that there is an $\epsilon > 0$ so that whenever $0 < x < \epsilon$, we have $f(x) > f(0)$.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Also assume that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty.$$

   (a) Prove that there is an x so that $f'(x) = 0$.
   (b) Prove that there is an x so that $f'(x) > 0$.
   (c) Prove that there is an x so that $f'(x) < 0$.
   (d) Given an example of a function f satisfying the statements above for which $-1 < f'(x) < 1$ for all $x \in \mathbb{R}$.

6. Let f be a differentiable function defined on $[\frac{1}{4}, \frac{3}{4}]$ so that $f(\frac{1}{4}) = f(\frac{3}{4})$. For each $a \in \mathbb{R}$, define $g_a(x) = ax(1 - x)$. We will prove that the graph of f is somewhere tangent to the graph of g_a for some a. (That is, there is an a and an $x \in (\frac{1}{4}, \frac{3}{4})$ so that $f(x) = g_a(x)$ and $f'(x) = g_a'(x)$.)
   (a) Define $h(x) = \frac{f(x)}{x(1-x)}$. Observe that $f(x) = g_a(x)$ whenever $a = h(x)$.
   (b) Show that there is an $x_0 \in (\frac{1}{4}, \frac{3}{4})$ with $h'(x_0) = 0$.
   (c) With x_0 as in part (b), show that there is an a so that $f(x_0) = g_a(x_0)$ and $f'(x_0) = g_a'(x_0)$.

7. Let $f(x) = x\sqrt{|x|}$. (This is the product of x and the square root of the absolute value of x.) At which points $x \in \mathbb{R}$ is f differentiable? Prove your answer is correct, and rigorously compute the derivative at all points at which f is differentiable.

8. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if for any $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all x, then f is strictly increasing.

9. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Complete the following definitions:
   (a) For any subset $S \subset [a, b]$,

   $M(f, S) =$

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   (b) A *partition* P of $[a, b]$ is ...
   (c) The *upper and lower Darboux sums* of f with respect the partition P are ...
   (d) The *upper and lower Darboux integrals* of f are ...
   (e) The function f is *integrable* on $[a, b]$ if ...
   (f) If f is integrable on $[a, b]$, the *value of the integral* of f over $[a, b]$ is ...

10. In class lectures and in the book, the following theorem is discussed:

    Theorem. Every monotonic function f on $[a, b]$ is integrable.

    Give a proof of this theorem in the case when f is increasing. You may use basic results about the Darboux integral which were established in class and in the book.

11. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \neq \frac{1}{2}$ and $f(\frac{1}{2}) = 5$. Use the definition of the Darboux integral to show that f is integrable and compute its integral.

12. Assume that $f : [0, 1] \to \mathbb{R}$ is a continuous function. Prove that f is integrable.

13. Assume that f and g are integrable functions on $[a, b]$ and that $f(x) \geq g(x)$ for all $x \in [a, b]$. Show that $\int_a^b f \geq \int_a^b g$.

14. Let f be a real-valued function which is differentiable on $[a, b]$. Let P be a partition of $[a, b]$. Prove that the Darboux sums of the derivative f' satisfy the inequality

    $$L(f', P) \leq f(b) - f(a) \leq U(f', P).$$

15. Let $f(x) = \cos(x)$.
    (a) State a version of Taylor's theorem.
    (b) Compute the Taylor series for f about zero.
    (c) Use Taylor's theorem to show that the Taylor series for f converges to f pointwise on $\mathbb{R}$. (You must use Taylor's theorem directly.)

16. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Suppose that $\lim_{x \to 1^+} f(x) = 7$ and $\lim_{x \to 1^-} f(x) = 3$. Define $F(x) = \int_0^x f(t)\, dt$.
    (a) Suppose $y > 1$. Write $F(y) - F(1)$ as an integral of f.
    (b) Show that there is a constant $c > 0$ so that $F(y) - F(1) \geq 6(y - 1)$ whenever $1 < y < 1 + c$.
    (c) Similarly, it follows that there is a $c > 0$ so that $F(1) - F(y) < 4(1 - y)$ whenever $1 - c < y < 1$. (You do not need to prove this.) Use these two facts to show that F is not differentiable at $x = 1$.