Math 323: Practice for Midterm 3
Solutions

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1. (a) Complete the ϵ-δ definition of continuity.
Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is **continuous** *at $x_0 \in dom(f)$ if and only if ...*

> **Solution:** for all $\epsilon > 0$ there is a $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0. \end{cases}$$

Use the ϵ-δ definition to prove that f is continuous at zero.

> **Solution:** Given any $\epsilon > 0$, choose $\delta = \min\{\epsilon^2, \sqrt{\epsilon}\}$. Suppose $|x - 0| < \delta$. Then, if $x \geq 0$ we have $|f(x)| = \sqrt{x} < \sqrt{\delta} \leq \sqrt{\epsilon^2} = \epsilon$. If $x < 0$ we have $|f(x)| = x^2 < \delta^2 \leq (\sqrt{\epsilon})^2 = \epsilon$. In either case we have $|f(x)| < \epsilon$ as desired.

2. (a) State the ϵ-δ definition of continuity of a function f at a point x_0.

> **Solution:** Look it up in the book.

(b) Use the ϵ-δ definition to directly prove that $f(x) = x + x^3$ is continuous at 0. (You may not use other results you know. Just verify the definition.)

> **Solution:** We just present the formal proof below.
> Given any $\epsilon > 0$ set $\delta = \min\{\frac{\epsilon}{2}, 1\}$. If $|x| < \delta$, then
>
> $$|x + x^3| = |x|(1 + x^2) < \delta(1 + \delta^2).$$
>
> Observe that $\delta \leq \frac{\epsilon}{2}$ and $1 + \delta^2 \leq 2$, so we have
>
> $$|x + x^3| < \delta(1 + \delta^2) \leq \frac{\epsilon}{2}(2) = \epsilon.$$
>
> This proves that $f(x)$ is continuous at zero.

3. (a) State the ϵ-δ definition of **continuity** of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in R$.

> **Solution:** The function f is **continuous** at x_0 if for all $\epsilon > 0$, there is a $\delta > 0$ so that $x \in \mathbb{R}$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

(b) Let f and g be real-valued continuous functions defined on $\mathbb{R}$. Let x_0 be a real number and suppose that $f(x_0) < g(x_0)$. Prove that there is an open interval (a, b) containing x_0 so that $f(x) < g(x)$ for every $x \in (a, b)$.

Solution: Define the function $h(x) = g(x) - f(x)$. Then $h(x)$ is continuous, because both g and f are continuous. Also $h(x_0) > 0$. Let $\epsilon = h(x_0)$. Then because h is continuous at x_0, we know that there is a δ so that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. The conclusion can also be written as

$$0 = f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon.$$

So we have shown that for all $x \in (x_0 - \delta, x_0 + \delta)$, we have $f(x) > 0$.

4. Let f and g be real-valued functions defined on $\mathbb{R}$. Suppose g is continuous at $x_0 \in \mathbb{R}$, and that f is continuous at $g(x_0)$. Use the ϵ-δ definition of continuity to prove that $h(x) = f \circ g(x)$ is continuous at x_0.

Solution: Fix $\epsilon > 0$. We will find a $\delta > 0$ so that $|x - x_0| < \delta$ implies that $|h(x) - h(x_0)| < \delta$.

Because f is continuous at $g(x_0)$, there is a $\delta_1 > 0$ so that $|y - g(x_0)| < \delta_1$ implies that $|f(y) - f \circ g(x_0)| < \epsilon$. Because g is continuous at x_0, there is an $\delta > 0$ so that $|x - x_0| < \delta$ implies that $|g(x) - g(x_0)| < \delta_1$.

Now fixing any x with $|x - x_0| < \delta$, we see that $|g(x) - g(x_0)| < \delta_1$, which in turn implies that $|f \circ g(x) - f \circ g(x_0)| < \epsilon$.

5. Give counterexamp les to the following false statements. You do not need to justify your answer.

(a) Every continuous real-valued function defined on $(0, 1)$ is uniformly continuous on $(0, 1)$.

Solution: The function $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous on $(0, 1)$.

(b) Every continuous real-valued function defined on $[0, \infty)$ is uniformly continuous on $[0, \infty)$.

Solution: The function $f(x) = x^2$ is continuous but not uniformly continuous on $[0, \infty)$.

(c) Let f be a real-valued function defined on $\mathbb{R}$. If the function $g(x) = |f(x)|$ is continuous, then $f(x)$ is continuous.

Solution: Let $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$

(d) If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Solution: Let $a_n = \frac{(-1)^n}{n}$. Then $\sum_{n=1}^{\infty} a_n$ converges by the alternating series test, but

$$\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}\frac{1}{n} = +\infty.$$

6. (a) State the Intermediate Value Theorem.

Solution: Let f be a continuous function on the interval $[a, b]$, and suppose that y lies between $f(a)$ and $f(b)$ (i.e. either $f(a) < y < f(b)$ or $f(b) < y < f(a)$). Then there is an $x \in (a, b)$ so that $f(x) = y$.

(b) Suppose $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous functions so that $f(x) \neq g(x)$ for all $x \in [0, 1]$. Prove that if $f(x_0) < g(x_0)$ for some $x_0 \in [0, 1]$, then $f(x) < g(x)$ for every $x \in [0, 1]$.

Solution: Assume to the contrary that there are such continuous functions f and g so that $f(x_0) < g(x_0)$ for some $x_0 \in [0, 1]$, but there is an $x \in [0, 1]$ so that $f(x) \geq g(x)$. Since $f(x) \neq g(x)$, we must have $f(x) > g(x)$. Let $h(x) = f(x) - g(x)$. This function is continuous because both f and g are continuous. We have $h(x_0) < 0$ and $h(x) > 0$, so by the Intermediate Value Theorem, there is a y between x_0 and x so that $h(y) = 0$. Then $f(x) = g(x)$, which is a contradiction.

7. Let f and g be real-valued functions defined on $(0, 1)$. Suppose that both f and g are uniformly continuous on $(0, 1)$. Prove that $h(x) = f(x)g(x)$ is uniformly continuous on $(0, 1)$.
(*Hint:* Consider the characterization of uniform continuity on a bounded set.)

Solution: Since f and g are uniformly continuous on $(0, 1)$, they have continuous extensions $\tilde{f}, \tilde{g} : [0, 1] \to \mathbb{R}$. That is, $\tilde{f}$ and $\tilde{g}$ are continuous functions so that $\tilde{f}(x) = f(x)$ and $\tilde{g}(x) = g(x)$ for every $x \in (0, 1)$. Then $\tilde{h}(x) = \tilde{f}(x)\tilde{g}(x)$ is a continuous function on $[0, 1]$ which extends $h(x) = f(x)g(x)$ on $(0, 1)$. We see that h is uniformly continuous on $(0, 1)$ because it has a continuous extension to $[0, 1]$.

8. (a) Complete the following definition.
Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is **uniformly continuous on** S *if . . .*

Solution: for all $\epsilon > 0$ there is a $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

(b) Use this definition to prove that the function $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

Solution: Here are two solutions.

Proof by contradiction: Suppose $f(x) = x^3$ was uniformly continuous on $\mathbb{R}$. Then for $\epsilon = 1$, there is a $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. Observe that $|f(x) - f(y)| = |x^3 - y^3| = |x - y||x^2 + xy + y^2|$. Letting $x = \frac{1}{\sqrt{\delta}}$ and $y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{2}$ we see $|x - y| = \frac{\delta}{2}$, $x^2 = \frac{1}{\delta}$, $xy > \frac{1}{\delta}$ and $y^2 > \frac{1}{\delta}$. So,

$$|f(x) - f(y)| \geq \frac{\delta}{2}(\frac{3}{\delta}) \geq \frac{3}{2}.$$

This is a contradiction to the statement that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$.

Proof by verifying the negative of the definition: We verify the negation of the definition. We need to show that there is an $\epsilon > 0$ so that for all $\delta > 0$ there is an $x, y \in \mathbb{R}$ so that $|x-y| < \delta$ and $|f(x)-f(y)| > \epsilon$. Set $\epsilon = 1$. Given any $\delta > 0$, consider the case when $y = x + \frac{\delta}{2}$. This satisfies $|x - y| = \frac{\delta}{2} < \delta$ and

$$|f(x) - f(x + y)| = |x^3 - (x + \frac{\delta}{2})^3)| = |\frac{3\delta}{2}x^2 + \frac{3\delta^2}{4}x + \frac{\delta^3}{8}|.$$

Observe that $\lim_{x\to\infty} |f(x) - f(x + \delta 2)| = +\infty$. So, there is an M so that $x > M$ implies $|f(x) - f(x + \delta 2)| > 2 > \epsilon = 1$.

9. (a) State the Intermediate Value Theorem.

Solution: If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$ [i.e., $f(a) < y < f(b)$ or $f(b) < y < f(a)$], there exists at least one $x \in (a, b)$ such that $f(x) = y$.

(b) Suppose that f is a polynomial of degree four of the form

$$f(x) = x^4 + ax^3 + bx^2 + cx + d,$$

with $a, b, c, d \in \mathbb{R}$. Also suppose there is a $y \in \mathbb{R}$ so that $f(y) < 0$. Prove that f has a real root. That is, prove that there is an $x_0 \in \mathbb{R}$ so that $f(x_0) = 0$.

Solution: We need to find a $z \in \mathbb{R}$ so that $f(z) > 0$. Then the intermediate value theorem will imply that there is zero of the polynomial between y and z.

Observe that

$$\lim_{x\to\infty} \frac{1}{x^4} f(x) = \lim_{x\to+\infty} 1 + \frac{a}{x} + \frac{b}{x^2} + \frac{x}{x^3} + \frac{d}{x^4} = 0,$$

since all the terms other than the first tend to zero as $x \to \infty$. Also note that $\lim_{x\to\infty} x^4 = +\infty$. Then,

$$\lim_{x\to\infty} f(x) = \big(\lim_{x\to\infty} x^4\big)\big(\lim_{x\to\infty} \frac{1}{x^4} f(x)\big) = +\infty.$$

This means that there is an M so that $z > M$ implies that $f(z) > 0$. Choose such a z. Then since $f(y) < 0 < f(z)$, by the intermediate value theorem there is an x between y and z so that $f(x) = 0$.

10. (a) State the Intermediate Value Theorem.

> **Solution:** Look it up in the book.

(b) Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous function so that $f(0) = 1$ and $f(1) = 0$. Prove that there is an $x \in (0, 1)$ so that $f(x) = x^2$.

> **Solution:** Let $g(x) = f(x) - x^2$. Then, $g(0) = 1$ and $g(1) = -1$. So by the Intermediate Value Theorem there is an $x \in (0, 1)$ so that $g(x) = 0$. For this x we have $f(x) = x^2$.

11. Let T_x be the triangle with sides of length $1 + x$, $2x + 1$, and $5 - 3x$. Prove that there is a $y \in [0, 1]$ so that

$$\text{Area}(T_y) \geq \text{Area}(T_x) \quad \text{for all } x \in [0, 1].$$

> **Solution:** The function $f : [0, 1] \to \mathbb{R}$ defined so that $f(x)$ is the area of T_x is a continuous function. (It is given by the semi-perimeter formula for area of a triangle.) Since $[0, 1]$ is compact, the function attains its maximum at some point $y \in [0, 1]$. This is the value of y needed.

12. Consider the following definitions:

Let f be a real-valued function defined on an open interval containing $a \in \mathbb{R}$. The function is *rightward increasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \geq f(a)$. Similarly, the function is *rightward decreasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \leq f(a)$.

(a) Prove that $f(x) = x - x^3$ is rightward increasing at $a = 0$.

> **Solution:** Let $\delta = 1$. We will show that $0 < x < 1$ implies that $f(x) \geq f(a) = 0$. Fix x with $0 < x < 1$. By multiplying by $x > 0$, we know that $0 < x^2 < x < 1$. Therefore by negation, we have $-1 < -x^2 < 0$. Then by adding one, $0 < 1 - x^2 < 1$. Now we multiply through by x again to obtain:
>
> $$0 < f(x) = x - x^3 < x < 1.$$
>
> In particular, $f(x) > 0$ as desired.

(b) Prove that the following function is neither rightward increasing nor rightward decreasing at $a = 0$:

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Solution: We will prove this by contradiction. Assume that the statement is false. Then f is either rightward increasing or rightward decreasing. We will consider each case separately.

Suppose f is rightward increasing. Then, there is a $\delta > 0$ so that $0 < x < \delta$ implies that $f(x) \geq f(a) = 0$. Fix this δ. Let $n \geq 1$ be an integer so that

$$n > \frac{\frac{1}{\delta} + \frac{\pi}{2}}{2\pi}.$$

Let $x = \frac{1}{2\pi n - \frac{\pi}{2}}$. Because of our choice of n, we have that $x > 0$ and

$$\frac{1}{x} = 2\pi n - \frac{\pi}{2} > 2\pi\left(\frac{\frac{1}{\delta} + \frac{\pi}{2}}{2\pi}\right) - \frac{\pi}{2} = \frac{1}{\delta}.$$

Therefore, $0 < x < \delta$ and by assumption it must be that $f(x) \geq 0$. But,

$$f(x) = x\sin(2\pi n - \frac{\pi}{2}) = -x < 0,$$

which contradicts this assumption.

Now suppose that f is rightward decreasing. Then, there is a $\delta > 0$ so that $0 < x < \delta$ implies that $f(x) \leq f(a) = 0$. Fix this δ. Let $n \geq 0$ be an integer so that

$$n > \frac{\frac{1}{\delta} - \frac{\pi}{2}}{2\pi}.$$

Let $x = \frac{1}{2\pi n + \frac{\pi}{2}}$.Because of our choice of n, we have that $x > 0$ and

$$\frac{1}{x} = 2\pi n + \frac{\pi}{2} > 2\pi\left(\frac{\frac{1}{\delta} - \frac{\pi}{2}}{2\pi}\right) + \frac{\pi}{2} = \frac{1}{\delta}.$$

Therefore, $0 < x < \delta$ and by assumption it must be that $f(x) \leq 0$. But,

$$f(x) = x\sin(2\pi n + \frac{\pi}{2}) = x > 0,$$

which contradicts this assumption.

13. There are two definitions of continuity at a point. (The book calls one the ϵ-δ definition of continuity but states it as a Theorem.) Complete the definition in these two ways.

    (a) The function f is continuous at x_0 in $\text{dom}(f)$ if, ...

    Solution: for every sequence (x_n) in $\text{dom}(f)$ converging to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

    (b) The function f is continuous at x_0 in $\text{dom}(f)$ if, ...

    Solution: for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is continuous and non-negative (i.e., $f(x) \geq 0$ for every $x \in \mathbb{R}$). Suppose $g : \mathbb{R} \to \mathbb{R}$ is another function which satisfies $|g(x)| \leq f(x)$ for every $x \in \mathbb{R}$. Prove that if $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then g is continuous at x_0.

Solution: We will assume that f and g are as described in the problem, and also that $f(x_0) = 0$. Observe that since $|g(x_0)| \leq f(x_0)$, we have $g(x_0) = 0$. We will prove that g is continuous at x_0 using the definition.

Using the sequence definition. Let (x_n) be an arbitrary sequence converging to x_0. We will prove that g is continuous by showing that $\lim g(x_n) = g(x_0)$. Since f is continuous at x_0, we see that $\lim f(x_n) = f(x_0) = 0$. Now recall that $|g(x)| \leq f(x)$. This means that

$$-f(x_n) \leq g(x_n) \leq f(x_n).$$

So, by the squeeze theorem, we see that

$$\lim g(x_n) = \lim f(x_n) = 0 = g(x_0),$$

which concludes the proof.

Using $\epsilon - \delta$ definition. Let $\epsilon > 0$. Since f is continuous at x_0, there is a $\delta > 0$ so that

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \epsilon.$$

Since $f(x_0) = 0$ and $f(x) \geq 0$, we see that $|f(x) - f(x_0)| = f(x)$. So, we see that $|x - x_0| < \delta$ implies $f(x) < \epsilon$. Then since $|g(x)| \leq f(x)$, we see that

$$|x - x_0| < \delta \quad \text{implies} \quad |g(x)| \leq f(x) < \epsilon.$$

Since $g(x_0) = 0$, this shows that $|x - x_0| < \delta$ implies $|g(x) - g(x_0)| < \epsilon$. So, g is continuous by definition.

14. Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.

Solution: Define the function $h : [a, b] \to \mathbb{R}$ by $h(x) = f(x) - g(x)$. Then, $h(a) \geq 0$ and $h(b) \leq 0$. If $h(a) = 0$, then $f(a) = g(a)$, and if $h(b) = 0$, then $f(b) = g(b)$. This proves the result in these cases. Otherwise, we have $h(a) > 0$ and $h(b) < 0$. Then by the intermediate value theorem, there is an $x_0 \in (a, b)$ so that $h(x_0) = 0$. For this value of x_0, we have $f(x_0) = g(x_0)$.

15. (a) State the ϵ-δ definition of **continuity** of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in R$.

Solution: The function f is **continuous** at x_0 if for all $\epsilon > 0$, there is a $\delta > 0$ so that $x \in \mathbb{R}$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

(b) Let f and g be real-valued continuous functions defined on $\mathbb{R}$. Let x_0 be a real number and suppose that $f(x_0) < g(x_0)$. Prove that there is an open interval (a, b) containing x_0 so that $f(x) < g(x)$ for every $x \in (a, b)$.

Solution: Define the function $h(x) = g(x) - f(x)$. Then $h(x)$ is continuous, because both g and f are continuous. Also $h(x_0) > 0$. Let $\epsilon = h(x_0)$. Then because h is continuous at x_0, we know that there is a δ so that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. The conclusion can also be written as

$$0 = f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon.$$

So we have shown that for all $x \in (x_0 - \delta, x_0 + \delta)$, we have $f(x) > 0$.

16. (a) Complete the following definition:
Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is *uniformly continuous* on S if ...

Solution: for each $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

(b) Let $a > 0$ be an arbitrary positive real number. Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on (a, ∞).

Solution: Let $\epsilon > 0$. We will show that for each $x > a$ and $y > a$, $|x - y| < a^2\epsilon$ implies $|f(x) - f(y)| < \epsilon$.
Let $x > a$ and $y > a$ be such that $|x - y| < a^2\epsilon$. We can assume without loss of generality that $x < y$. Then $0 < y - x < a^2\epsilon$. We have

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = \frac{y - x}{xy} < \frac{a^2\epsilon}{a^2} = \epsilon.$$

17. Recall that if $g : \mathbb{R} \to \mathbb{R}$ is a function and $A \subset \mathbb{R}$, then the *image* of A under g is

$$g(A) = \{g(a) : \ a \in A\}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function.

(a) Prove that if A is a bounded non-empty set, then $\sup f(A) \leq f(\sup A)$.

Solution: Let A be a bounded non-empty set. Then by the completeness axiom, the set has a least upper bound $\sup A \in \mathbb{R}$. By definition of upper bound, $a \leq \sup A$ for all $a \in A$. Since f is increasing, $f(a) \leq f(\sup A)$ for all $a \in A$. Thus, $f(\sup A)$ is an upper bound for $f(A)$. But an upper bound is always non-strictly larger than the least upper bound, so $\sup f(A) \leq f(\sup A)$.

(b) Prove that if f is also continuous, then $\sup f(A) = f(\sup A)$.

Solution: Let $a_n \in A$ be a sequence of elements of A converging to $\sup A$. Then, by continuity, $\lim f(a_n) = f(\sup A)$. In particular, we see that any upper bound for $f(A)$ must be larger than $f(\sup A)$ (since it must be larger than each $f(a_n)$). In particular, the least upper bound of $f(A)$ must satisfy this, so $\sup f(A) \geq f(\sup A)$. Combining this inequality with the one from part (a) yields $\sup f(A) = f(\sup A)$.

Remark: You are expected to know that in any set there is a sequence of elements approaching the least upper bound. This sequence can be produced by induction since $\sup(A) - \frac{1}{n}$ is not an upper bound for any n.