Math 323: Practice for Midterm 3

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §15-§19, not including starred sections. Earlier sections may also be covered. You are expected to know all material covered in the course up until now.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

absolutely convergent, alternating series, continuous, continuous at a point, continuous on a set, discontinuous, composition, bounded function, uniformly continuous, strictly decreasing, decreasing, strictly increasing, increasing.

Theorems. Theorems given names in the book are often the most important. Theorems (and similar results) you may be required to state:

15.2: Integral tests, 15.3: Alternating Series Theorem, 17.2: δ-ϵ definition of continuity, 18.1: Continuous functions on closed and bounded sets attain their maximum and minimum, 18.2: Intermediate Value Theorem

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test.

1. (a) Complete the ϵ-δ definition of continuity.
   Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is **continuous** *at $x_0 \in dom(f)$ if and only if ...*

   (b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0. \end{cases}$$

   Use the ϵ-δ definition to prove that f is continuous at zero.

2. (a) State the ϵ-δ definition of continuity of a function f at a point x_0.

   (b) Use the ϵ-δ definition to directly prove that $f(x) = x + x^3$ is continuous at 0. (You may not use other results you know. Just verify the definition.)

3. (a) State the ϵ-δ definition of **continuity** of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in R$.

   (b) Let f and g be real-valued continuous functions defined on $\mathbb{R}$. Let x_0 be a real number and suppose that $f(x_0) < g(x_0)$. Prove that there is an open interval (a, b) containing x_0 so that $f(x) < g(x)$ for every $x \in (a, b)$.

4. Let f and g be real-valued functions defined on $\mathbb{R}$. Suppose g is continuous at $x_0 \in \mathbb{R}$, and that f is continuous at $g(x_0)$. Use the ϵ-δ definition of continuity to prove that $h(x) = f \circ g(x)$ is continuous at x_0.

5. Give counterexamp les to the following false statements. You do not need to justify your answer.
   (a) Every continuous real-valued function defined on $(0, 1)$ is uniformly continuous on $(0, 1)$.
   (b) Every continuous real-valued function defined on $[0, \infty)$ is uniformly continuous on $[0, \infty)$.
   (c) Let f be a real-valued function defined on $\mathbb{R}$. If the function $g(x) = |f(x)|$ is continuous, then $f(x)$ is continuous.
   (d) If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

6. (a) State the Intermediate Value Theorem.
   (b) Suppose $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous functions so that $f(x) \neq g(x)$ for all $x \in [0, 1]$. Prove that if $f(x_0) < g(x_0)$ for some $x_0 \in [0, 1]$, then $f(x) < g(x)$ for every $x \in [0, 1]$.

7. Let f and g be real-valued functions defined on $(0, 1)$. Suppose that both f and g are uniformly continuous on $(0, 1)$. Prove that $h(x) = f(x)g(x)$ is uniformly continuous on $(0, 1)$.
   (*Hint:* Consider the characterization of uniform continuity on a bounded set.)

8. (a) Complete the following definition.
   Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is **uniformly continuous on** S *if . . .*
   (b) Use this definition to prove that the function $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

9. (a) State the Intermediate Value Theorem.
   (b) Suppose that f is a polynomial of degree four of the form
   $$f(x) = x^4 + ax^3 + bx^2 + cx + d,$$
   with $a, b, c, d \in \mathbb{R}$. Also suppose there is a $y \in \mathbb{R}$ so that $f(y) < 0$. Prove that f has a real root. That is, prove that there is an $x_0 \in \mathbb{R}$ so that $f(x_0) = 0$.

10. (a) State the Intermediate Value Theorem.
    (b) Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous function so that $f(0) = 1$ and $f(1) = 0$. Prove that there is an $x \in (0, 1)$ so that $f(x) = x^2$.

11. Let T_x be the triangle with sides of length $1 + x$, $2x + 1$, and $5 - 3x$. Prove that there is a $y \in [0, 1]$ so that
    $$\text{Area}(T_y) \geq \text{Area}(T_x) \quad \text{for all } x \in [0, 1].$$

12. Consider the following definitions:

    Let f be a real-valued function defined on an open interval containing $a \in \mathbb{R}$. The function is *rightward increasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \geq f(a)$. Similarly, the function is *rightward decreasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \leq f(a)$.

(a) Prove that $f(x) = x - x^3$ is rightward increasing at $a = 0$.

(b) Prove that the following function is neither rightward increasing nor rightward decreasing at $a = 0$:

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

13. There are two definitions of continuity at a point. (The book calls one the ϵ-δ definition of continuity but states it as a Theorem.) Complete the definition in these two ways.

   (a) The function f is continuous at x_0 in $\text{dom}(f)$ if, ...

   (b) The function f is continuous at x_0 in $\text{dom}(f)$ if, ...

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is continuous and non-negative (i.e., $f(x) \geq 0$ for every $x \in \mathbb{R}$). Suppose $g : \mathbb{R} \to \mathbb{R}$ is another function which satisfies $|g(x)| \leq f(x)$ for every $x \in \mathbb{R}$. Prove that if $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then g is continuous at x_0.

14. Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.

15. (a) State the ϵ-δ definition of **continuity** of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in R$.

   (b) Let f and g be real-valued continuous functions defined on $\mathbb{R}$. Let x_0 be a real number and suppose that $f(x_0) < g(x_0)$. Prove that there is an open interval (a, b) containing x_0 so that $f(x) < g(x)$ for every $x \in (a, b)$.

16. (a) Complete the following definition:
   Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is *uniformly continuous* on S if ...

   (b) Let $a > 0$ be an arbitrary positive real number. Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on (a, ∞).

17. Recall that if $g : \mathbb{R} \to \mathbb{R}$ is a function and $A \subset \mathbb{R}$, then the *image* of A under g is

$$g(A) = \{g(a) : \; a \in A\}.$$

   Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function.

   (a) Prove that if A is a bounded non-empty set, then $\sup f(A) \leq f(\sup A)$.

   (b) Prove that if f is also continuous, then $\sup f(A) = f(\sup A)$.