Math 323: Practice for Midterm 2

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §9-§14, not including starred sections. Earlier sections may also be covered especially material in §7-8, which is basic material about sequences. You are expected to know all material covered in the course up until now.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

bounded sequence, nonincreasing sequence, nondecreasing sequence, monotone sequence, Cauchy sequence, subsequence, subsequential limit, limit superior ($\limsup$*), and limit inferior (*$\liminf$*), Cauchy criterion, absolutely convergent*

Theorems. Theorems given names in the book are often the most important. Theorems (and similar results) you may be required to state:

11.5: Bolzano-Weierstrass Theorem, 14.6 Comparison Test, 14.8: Ratio Test, 14.9: Root Test

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test.

1. Suppose $0 < p < 1$. Consider the sequence defined inductively by $s_1 = 1$, and $s_{n+1} = ps_n + 1$ for $n \in \mathbb{N}$.
   (a) Use induction to prove that $s_n > 0$ for all $n \in \mathbb{N}$.
   (b) Prove that $\frac{1}{1-p}$ is an upper bound for (s_n).
   (c) Use this upper bound to prove that (s_n) is increasing.
   (d) Can you conclude that (s_n) converges? Why or why not? If it does converge, what is the limit? Why?
2. Suppose (s_n) is a sequence of real numbers such that $s_n < 0$ when n is odd, and $s_n > 1$ when n is even. Prove that the sequence (s_n) does not converge to a real number by proving that (s_n) is not Cauchy.
3. Suppose (s_n) is a sequence of real numbers with the property that $|s_n| > |s_{n+1}|$ for all $n \in \mathbb{N}$.
   (a) Prove that s_n has a subsequence which converges to a real number.
   (b) Explain why $\lim_{n\to\infty} |s_n|$ exists and is a real number.
   (c) Must $\lim_{n\to\infty} s_n$ exist? If it must exist, prove the limit exists. Otherwise, find an example of a sequence (s_n) which does not converge, but has the property that $|s_n| > |s_{n+1}|$ for all $n \in \mathbb{N}$.
4. (a) Complete the following definition:
   A sequence (s_n) of real numbers is said to converge to the real number s provided that...

(b) Use the definition to prove the following result:
Suppose (s_n) and (t_n) are sequences of real numbers. If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$.

5. Give examples of the following.
   (a) A bounded sequence which does not converge.
   (b) A monotone sequence which diverges.
   (c) A sequence (s_n) such that the set of subsequential limits is $\{-\infty, +\infty\}$.

6. Define the sequence (s_n) inductively by $s_1 = 1$, and $s_{n+1} = \frac{1}{s_n} + \frac{1}{n}$.
   (a) Let $n_k = 2k + 1$ for $k \in \mathbb{N}$. Then $\{n_k\}$ is the odd natural numbers. What can you say about the subsequence $t_k = s_{n_k}$?
   (b) Prove that $s_n \to 1$ as $n \to \infty$. (You may use the definition, or limit theorems.)

7. Suppose that (a_n) and (b_n) are sequences satisfying $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

8. Suppose that (a_n) is a sequence which satisfies the following two statements.
   1. $|a_n + a_{n+1}| < \frac{1}{2^n}$ for all $n \in \mathbb{N}$.
   2. $\lim_{n\to\infty} a_n = 0$.

   (a) Prove that $\sum_{n=1}^{\infty} a_n$ converges.
   (b) Give an example of a sequence (a_n) so that $\sum_{n=1}^{\infty} a_n$ does not converge, but (a_n) still satisfies statement 1 above.

9. Recall that a *counterexample* is an example which proves that a statement is false. Give counterexamples to the following false statements.
   (a) Every convergent series is absolutely convergent.
   (b) If (a_n) is a monotone sequence which converges to zero, then the series $\sum_{n=1}^{\infty} a_n$ converges.

10. (a) State the **Cauchy criterion** for convergence of the series $\sum_{n=1}^{\infty} a_n$.
    (b) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} a_n = 0$.

11. Define the sequence $s_n = \sin(n)$.
    (a) Prove that the sequence (s_n) has a convergent subsequence.
    (b) Prove that this sequence has no subsequence (s_{n_k}) which converges to 2.

12. For each of the following properties, either give an example of a sequence satisfying the property or explain why such a sequence is impossible.
    (a) A bounded sequence which does not converge to a real number.
    (b) An unbounded sequence with a subsequence which converges to zero.
    (c) A sequence whose set of subsequential limits is exactly $S = \{\frac{1}{n} \ : \ n \in \mathbb{N}\}$.

13. Determine if the series $\sum_{n=1}^{\infty} \frac{n^4}{2^n}$ converges. Justify your answer. (You may use any theorem you like.)

14. Let (a_n) be a sequence of real numbers which is both non-decreasing and bounded. Give the proof that the sequence converges.

15. State whether each of the following statements is true or false. If the statement is true, briefly explain why. If the statement is false, give a counterexample. Your explanations and counterexamples need not be more than a sentence or two.

    (a) If (a_n) is any sequence of real numbers and the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

    (b) If (a_n) is any sequence so that $\lim a_n = 0$, then $\sum a_n$ converges.

    (c) If (a_n) is any sequence of positive numbers and $\lim \frac{a_{n+1}}{a_n} = 1$, then $\sum a_n$ converges.

    (d) If (a_n) is any sequence of positive numbers and $\sum a_n$ converges, then $\sum a_n^2$ converges.

16. (a) State the Bolzano-Weierstrass Theorem.

    (b) Let (s_n) be a sequence of real numbers so that there are infinitely many $n \in \mathbb{N}$ with $|s_n| < 1$. Prove that (s_n) has a convergent subsequence.

17. (a) Suppose $S \subset \mathbb{R}$ and $\sup S = M$. Prove that if $N < M$ then there is an $s \in S$ so that $s > N$.

    (b) Suppose (s_n) is a sequence and $\limsup s_n = x \in \mathbb{R}$. Prove that if $y < x$ then there is an n so that $s_n > y$.

18. (a) Suppose (a_n) and (b_n) are sequences with $\limsup a_n = a \in \mathbb{R}$ and $\limsup b_n = b \in \mathbb{R}$. Prove that $\limsup a_n + b_n \leq a + b$.

    (b) Give an example of sequences (a_n) and (b_n) where

$$\limsup(a_n + b_n) < \limsup a_n + \limsup b_n.$$