Math 323: Advanced Calculus I: Third Midterm Solutions

Thursday, April 6th, 2017

Prof. Hooper

1. (a) (8 points) *Complete the following definition:*
   Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is *continuous* at x_0 in $\text{dom}(f)$ if ...

   Solution: for every sequence (x_n) in $\text{dom}(f)$ converging to x_0, we have $\lim_n f(x_n) = f(x_0)$.

   (b) (8 points) *Complete the ϵ-δ definition of continuity (which in our text appears as a Theorem):*
   Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is *continuous* at x_0 in $\text{dom}(f)$ if and only if ...

   Solution: for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x)f(x_0)| < \epsilon$.

   (c) (8 points) *Complete the following definition:*
   Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is *uniformly continuous* on S if ...

   Solution: for each $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

   (d) (8 points) *State the Intermediate Value Theorem.*

   Solution: If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$ [i.e., $f(a) < y < f(b)$ or $f(b) < y < f(a)$], there exists at least one x in (a, b) such that $f(x) = y$.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by the following rule

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For the following questions, you may use any definition of continuity that you wish.

(a) (12 points) Prove that f is continuous at zero.

> **Solution:** We will use the ϵ-δ definition of continuity. Observe that $f(0) = 0$. Let $\epsilon > 0$ be arbitrary. We will show that $|x-0| < \epsilon$ implies $|f(x)-0| < \epsilon$. (That is, we can set $\delta = \epsilon$.) Suppose $|x| < \epsilon$. Then $f(x) = \pm x$ and so $|f(x)| = |x| < \epsilon$.

(b) (12 points) Prove that f is not continuous at one.

> **Solution:** We will use the sequence definition of continuity. Suppose to the contrary that f is continuous at one. Then for any sequence (x_n) so that $\lim x_n = 1$ we have $\lim f(x_n) = f(1) = 1$. Recall that $\sqrt{2}$ is irrational. Define $x_n = 1 + \frac{\sqrt{2}}{n}$. Then and $\lim x_n = 1$. Since each $x_n \notin \mathbb{Q}$ we have
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> $$\lim f(x_n) = \lim -x_n = -\lim x_n = -1.$$
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> This is a contradiction since $\lim x_n = 1$ but $\lim f(x_n) \neq f(1)$.

3. State if the following statements are true or false. If true, briefly explain why. If false, give a counterexample.

(a) (6 points) A continuous real-valued function whose domain is a bounded interval is a bounded function.

> **Solution:** False. The function $f : (0,1) \to \mathbb{R}$ defined by $f(x) = 1/x$ is not bounded.

(b) (6 points) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous then so is the function

$$h(x) = f(x)g(x) + f\big(g(x)\big).$$

> **Solution:** True. Since f is continuous and g is continuous, so is $f \cdot g$ and $f \circ g$. Since these are both continuous so is their sum. These were theorems about continuity.

(c) (6 points) If the function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on each interval of the form $[n, n+1]$ with $n \in \mathbb{Z}$, then it is uniformly continuous on $\mathbb{R}$.

> **Solution:** False. Recall that a continuous function on a closed and bounded function is always uniformly continuous. Also recall that $f(x) = x^2$ is not uniformly continuous. So, $f(x) = x^2$ is a counterexample.

(d) (6 points) If $\sum_{n=0}^{\infty} a_n$ converges then so does $\sum_{n=0}^{\infty} |a_n|$.

> **Solution:** False. The series $\sum \frac{(-1)^n}{n}$ converges because it is alternating, but $\sum \frac{1}{n}$ diverges by the integral test.

(e) (6 points) A uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ attains its maximum. (The book says "assumes its maximum" for the same notion.)

> **Solution:** False. The function $f(x) = x$ is uniformly continuous but does not attain a maximum. (Recall that f attains its maximum if there is an x_0 in the domain so that $f(x_0) \geq f(x)$ for every point x in the domain.)

4. (14 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to R$ are continuous functions. Prove that the composition $g \circ f$ is continuous.
(*Remark:* I am asking you to prove a basic result about continuity, so you should only one use the definitions of continuity to prove this. Either definition is fine.)

> **Solution:**
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> **ϵ-δ proof.** Let $x_0 \in \mathbb{R}$. We will show that $g \circ f$ is continuous at x_0. Let $\epsilon > 0$. We must show that there is a $\delta > 0$ so that $|x - x_0| < \delta$ implies $|g \circ f(x) - g \circ f(x_0)| < \epsilon$.
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> (1) Since g is continuous, there is a $\delta_g > 0$ so that $|y - f(x_0)| < \delta_g$ implies $|g(y) - g \circ f(x_0)| < \epsilon$.
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> (2) Since f is continuous, there is a $\delta > 0$ so that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \delta_g$.
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> Now suppose $x \in \mathbb{R}$ and $|x - x_0| < \delta$. Then by (2) we know $|f(x) - f(x_0)| < \delta_g$. By (1) with $y = f(x)$ we see $|g \circ f(x) - g \circ f(x_0)| < \epsilon$.
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> **Sequence proof.** Let $x_0 \in \mathbb{R}$. We will show that $g \circ f$ is continuous at x_0. Let x_n be any sequence in $\mathbb{R}$ so that $\lim x_n = x_0$. We must show that $\lim g \circ f(x) = g \circ f(x_0)$. Since f is continuous an $\lim x_n = x_0$ we know that $\lim f(x_n) = f(x_0)$. Since g is continuous and $\lim f(x_n) = f(x_0)$ we know that $\lim g\big(f(x_n)\big) = g\big(f(x_0)\big)$.