Math 346: Linear Algebra: Take Home Quiz 9

Solutions:

1. Suppose A is a 4×4 matrix whose characteristic polynomial is $\lambda^4 + 1$.

   (a) (4 points) What are all the eigenvalues of A? (*Hint:* They are complex!)

   Solution: Observe that the roots satisfy $\lambda^4 = -1$ so we are looking for 4-th roots of -1. We can write -1 as $-1 = 1(\cos\pi + i\sin\pi)$. Recall that if $\lambda = r(\cos\theta + i\sin\theta)$ with $r \geq 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ then

   $$\lambda^4 = r^4\big(\cos(4\theta) + i\sin(4\theta)\big).$$

   So, we would have $r = 1$ and $4\theta = \pi$. This gives $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{4}$ or $\theta = \frac{7\pi}{4}$. These give four solutions:

   $$\lambda_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},$$

   $$\lambda_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},$$

   $$\lambda_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2},$$

   $$\lambda_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

   (b) (4 points) Is A diagonalizable? Possible answers include: A must be diagonalizable, A is never diagonalizable, or A is sometimes diagonalizable and sometimes not diagonalizable. Explain why your answer is correct.

   Solution: Yes it must be diagonalizable because there are four distinct eigenvalues. Each eigenvalue corresponds to at least one eigenvector, and eigenvectors with distinct eigenvectors are linearly independent. Therefore four eigenvectors (taken one for each eigenvalue) must form a basis for $\mathbb{C}^4$, and give the columns for an invertible matrix satisfying $A = X\Lambda X^{-1}$ where Λ is the diagonal matrix with λ_1, λ_2, λ_3 and λ_4 on the diagonal.

2. (10 points) Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

   (a) (6 points) Find eigenvalues and eigenvectors of A forming a basis for $\mathbb{R}^3$.

Solution: First we will compute the characteristic polynomial $\det(A-\lambda I)$. We have

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3+2+2-(1-\lambda)-4(1-\lambda)-(1-\lambda).$$

Simplifying we see

$$\det(A-\lambda I) = -\lambda^3 + 3\,\lambda^2 + 3\,\lambda - 1.$$

We should check for roots of the form ± 1 since integer roots should divide the constant term (which is -1). Observe that -1 is a root. Set $\lambda_1 = -1$. This means we can factor $\lambda + 1$ out. We see

$$\det(A-\lambda I) = -\big(\lambda^2 - 4\,\lambda + 1\big)(\lambda+1).$$

To find the other roots we apply the quadratic formula. We have

$$\lambda = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}.$$

The other two eigenvalues are $\lambda_2 = 2+\sqrt{3}$ and $\lambda_3 = 2-\sqrt{3}$.
Now we will find an eigenvector associated to $\lambda_1 = -1$. It must solve $(A+I)\mathbf{x}_1 = \mathbf{0}$. We have

$$\begin{aligned} A+I &= \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore we see that there is an eigenvector $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Now we will find an eigenvector associated to $\lambda_2 = 2+\sqrt{3}$. It must solve $(A-\lambda_2 I)\mathbf{x}_1 = \mathbf{0}$. We have

$$A - \lambda_2 I = \begin{pmatrix} -1-\sqrt{3} & 1 & 2 \\ 1 & -1-\sqrt{3} & 1 \\ 2 & 1 & -1-\sqrt{3} \end{pmatrix}$$

Switching the first two rows (to get a 1 in the upper left) yields

$$A - \lambda_2 I \sim \begin{pmatrix} 1 & -1-\sqrt{3} & 1 \\ -1-\sqrt{3} & 1 & 2 \\ 2 & 1 & -1-\sqrt{3} \end{pmatrix}.$$

By adding multiples $1+\sqrt{3}$ times row 1 to two subtracting twice row 1 from row 3 we see

$$A-\lambda_2 I \sim \begin{pmatrix} 1 & -1-\sqrt{3} & 1 \\ 0 & -3-2\sqrt{3} & 3+\sqrt{3} \\ 0 & 3+2\sqrt{3} & -3-\sqrt{3} \end{pmatrix}.$$

Observe row 3 is the negation of row 2. Thus adding row 2 to row 3 yields

$$A-\lambda_2 I \sim \begin{pmatrix} 1 & -1-\sqrt{3} & 1 \\ 0 & -3-2\sqrt{3} & 3+\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}.$$

To get the square root out of the pivot in the center we multiply row 2 by $-3+2\sqrt{3}$ (which is the algebraic conjugate of the entry $-3-2\sqrt{3}$. We have

$$A-\lambda_2 I \sim \begin{pmatrix} 1 & -1-\sqrt{3} & 1 \\ 0 & -3 & -3+3\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we divide the second row by -3:

$$A-\lambda_2 I \sim \begin{pmatrix} 1 & -1-\sqrt{3} & 1 \\ 0 & 1 & 1-\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}.$$

To get to reduced echelon form we add $1+\sqrt{3}$ times row 2 to row 1:

$$A-\lambda_2 I \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1-\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives an eigenvector of $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1+\sqrt{3} \\ 1 \end{pmatrix}$.

The eigenvalue $\lambda_3 = 2-\sqrt{3}$ is the algebraic conjugate of $\lambda_2 = 2+\sqrt{3}$. It follows that $\mathbf{x}_3$ will be the algebraic conjuate of $\mathbf{x}_2$, so $\mathbf{x}_3 = \begin{pmatrix} 1 \\ -1-\sqrt{3} \\ 1 \end{pmatrix}$. We can check that this works by comparing $A\mathbf{x}_3$ to $\lambda_3\mathbf{x}_3$. We have

$$A\mathbf{x}_3 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1-\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 2-\sqrt{3} \\ 1-\sqrt{3} \\ 2-\sqrt{3} \end{pmatrix}.$$

On the other hand

$$\lambda_3\mathbf{x}_3 = (2-\sqrt{3}) \begin{pmatrix} 1 \\ -1-\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 2-\sqrt{3} \\ 1-\sqrt{3} \\ 2-\sqrt{3} \end{pmatrix}.$$

It worked.

(b) (2 points) Diagonalize A. (Find a matrix X and a diagonal matrix Λ so that $A = X\Lambda X^{-1}$.)

Solution: From the previous part, we have $A = X\Lambda X^{-1}$ where

$$X = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1+\sqrt{3} & -1-\sqrt{3} \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{pmatrix}.$$

(c) (2 points) Check that the eigenvectors in the basis of A are orthogonal.

Solution: We must check that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$, $\mathbf{x}_1 \cdot \mathbf{x}_3 = 0$ and $\mathbf{x}_2 \cdot \mathbf{x}_3 = 0$. We have:

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1+\sqrt{3} \\ 1 \end{pmatrix} = 0.$$

$$\mathbf{x}_1 \cdot \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1-\sqrt{3} \\ 1 \end{pmatrix} = 0$$

$$\mathbf{x}_2 \cdot \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1+\sqrt{3} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1-\sqrt{3} \\ 1 \end{pmatrix} = 1 - 2 + 1 = 0.$$