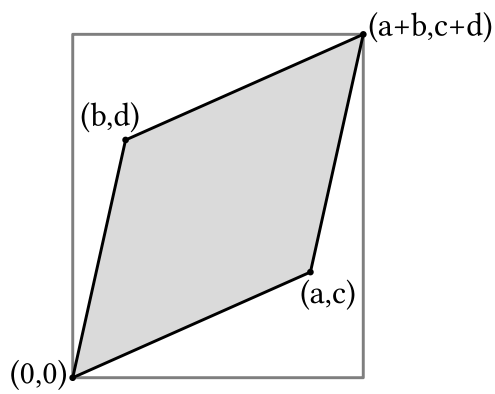
Math 346: Linear Algebra: Take Home Quiz 7

Solutions:

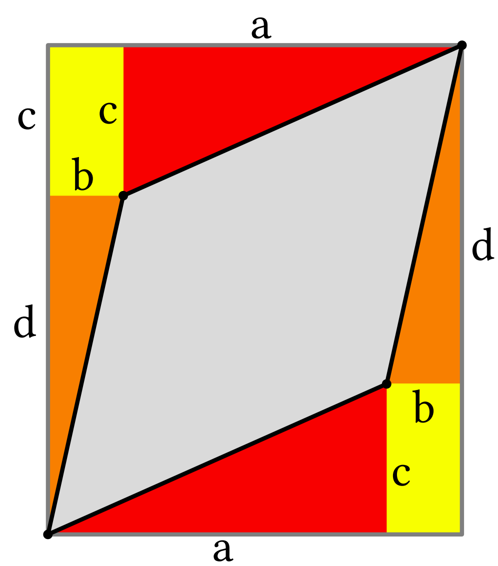
1. (10 points) Let P be the parallelogram in $\mathbb{R}^2$ so that the edge vectors are (a, c) and (b, d). Also assume that $a > b$ and $d > c$ so that P looks as in the figure below. Use elementary geometry formulas (e.g., the area of a triangle is half the base times the height) to show that the area of P is $ad - bc$, which is the same as the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. (*Hint:* The parallelogram lives inside a rectangle as in the figure below. What is the area of the region obtained by removing the parallelogram from the rectangle?)



Solution: Let R be the rectangle in the figure. Since the area of a rectangle is its base times its height we know

$$\text{Area}(R) = (a + b)(c + d) = ac + bc + ad + bd.$$

The region R with the parallelogram removed can be cut into two rectangles, and four right triangles. These regions have been color coded below and the lengths of horizontal and vertical edges have been labeled.



Observe that the yellow rectangles each have area bc, the red triangles each have area $\frac{ac}{2}$, and the orange triangles each have area $\frac{bd}{2}$. So we have

$$\text{Area}(R \setminus P) = 2(bc) + 2(\frac{ac}{2}) + 2(\frac{bd}{2}) = 2bc + ac + bd.$$

Finally, we compute

$$\text{Area}(P) = \text{Area}(R) - \text{Area}(R \setminus P) = (ac + bc + ad + bd) - (2bc + ac + bd) = ad - bc.$$

2. (10 points) Find an orthonormal basis for the column space of

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{pmatrix}.$$

Solution: Since the columns are not multiples of one another, it is clear that the columns are linearly independent and $C(A)$ is a plane. A basis is then given by the two column vectors

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We will apply the Gram-Schmidt algorithm to this basis. Observe that the length of

$\mathbf{a}_1$ is 3. We define

$$\mathbf{q}_1 = \frac{1}{\|\mathbf{a}_1\|}\mathbf{a}_1 = \frac{1}{3}\mathbf{a}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

This is the first element of our basis. Now let $\hat{\mathbf{a}}_2$ be the projection of $\mathbf{a}_2$ to *span* $\{\mathbf{q}_1\}$. We have

$$\hat{\mathbf{a}}_2 = (\mathbf{q}_1 \cdot \mathbf{a}_2)\mathbf{q}_1 = \frac{5}{3}\mathbf{q}_1 = \frac{5}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

We compute

$$\mathbf{a}_2 - \hat{\mathbf{a}}_2 = \frac{1}{9}\left[\begin{pmatrix} 9 \\ 9 \\ 9 \end{pmatrix} - \begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix}\right] = \frac{1}{9}\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}.$$

The length of $\mathbf{a}_2 - \hat{\mathbf{a}}_2$ is $\frac{3\sqrt{2}}{9}$. Then we define

$$\mathbf{q}_2 = \frac{1}{\|\mathbf{a}_2 - \hat{\mathbf{a}}_2\|}(\mathbf{a}_2 - \hat{\mathbf{a}}_2) = \frac{9}{3\sqrt{2}} \cdot \frac{1}{9}\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} = \frac{\sqrt{2}}{6}\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}.$$

Our orthonormal basis is

$$\{\mathbf{q}_1, \mathbf{q}_2\} = \left\{\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \frac{\sqrt{2}}{6}\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}\right\}.$$