Math 346: Linear Algebra: Take Home Quiz 6

Solutions:

1. Let $V \subset \mathbb{R}^3$ denote the subspace of vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that $x - y + z = 0$.

   (a) (3 points) Find bases for both V and its orthogonal complement $V^\perp$.

   Solution: Let $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Then V is the null space of $\mathbf{n}^T$ (i.e., $V = N(\mathbf{n}^T)$). Then by the Fundamental Theorem of Linear Algebra II, $V^\perp = C(\mathbf{n}) = \mathit{span}\{\mathbf{n}\}$. Thus $\{\mathbf{n}\}$ is a basis for $V^\perp$.

   To find a basis for V we apply our algorithm to find a basis for a null space. The matrix $\mathbf{n}^T$ is already in reduced echelon form and corresponds to the equation $x - y + z = 0$. We have $x = y - z$ and y and z are free variables. Thus a basis for V is given by

   $$\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}.$$

   (b) (4 points) Compute 3×3 matrices M and N so that for any $\mathbf{x} \in \mathbb{R}^3$, $M\mathbf{x}$ is the projection of $\mathbf{x}$ to V and $N\mathbf{x}$ is the projection of $\mathbf{x}$ to $V^\perp$.

   Solution: Since $\{\mathbf{n}\}$ is a basis for $V^\perp$, the projection matrix is given by

   $$N = \frac{1}{\|\mathbf{n}\|^2}\mathbf{n}\mathbf{n}^T = \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

   To find the projection matrix M to V, define A to be the matrix where the basis vectors for V are the columns:

   $$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Then from the general formula for the projection matrix, $M = A(A^TA)^{-1}A^T$. By a computation we see that

   $$M = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

(c) (3 points) Show that for any $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} = M\mathbf{x} + N\mathbf{x}$.

Solution: Observe by a simple calculation that $M + N = I$. Then for any $\mathbf{x} \in \mathbb{R}^3$ we have

$$M\mathbf{x} + N\mathbf{x} = (M + N)\mathbf{x} = I\mathbf{x} = \mathbf{x}.$$

2. (10 points) Now let $V \subset \mathbb{R}^n$ be a subspace and M be the $n \times n$ matrix so that for any $\mathbf{x} \in \mathbb{R}^n$, $M\mathbf{x}$ is the projection of $\mathbf{x}$ to V. Prove that $I - M$ is the matrix defining the projection of $\mathbf{x}$ onto $V^\perp$. (Use what you know about projections and the definition of orthogonal complement.)

Solution: We discussed in class that the projection of $\mathbf{x}$ to V is the unique vector $\hat{\mathbf{x}} \in V$ so that $x - \hat{\mathbf{x}}$ is perpendicular to (every vector in) V.

Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary and $\hat{x} = Mx$ be the projection of $\mathbf{x}$ to V. From the fact above $\mathbf{x} - \hat{\mathbf{x}}$ is perpendicular to every vector in V. Thus $\mathbf{x} - \hat{\mathbf{x}} \in V^\perp$. We claim that $\mathbf{x} - \hat{\mathbf{x}}$ is the projection of $\mathbf{x}$ to $V^\perp$. To prove this we use the fact above again. From the fact (with V replaced by $V^\perp$) we know that $\mathbf{y} \in V^\perp$ is the projection of $\mathbf{x}$ to $V^\perp$ if and only if $\mathbf{x} - \mathbf{y}$ is perpendicular to every vector in $V^\perp$. Now observe that in the case of $\mathbf{y} = \mathbf{x} - \hat{\mathbf{x}}$ we have $\mathbf{x} - \mathbf{y} = \hat{\mathbf{x}}$ and $\hat{\mathbf{x}} \in V$ so by definition of $V^\perp$ we know that every vector in $V^\perp$ is perpendicular to $\hat{\mathbf{x}}$. Thus we have shown $\mathbf{y} = \mathbf{x} - \hat{\mathbf{x}}$ is the projection of $\mathbf{x}$ to $V^\perp$.

It remains to deal with the matrices. We have $\hat{x} = Mx$ so for any $\mathbf{x}$ as above we have

$$\mathbf{y} = \mathbf{x} - \hat{x} = \mathbf{x} - M\mathbf{x} = I\mathbf{x} - M\mathbf{x} = (I - M)\mathbf{x}.$$

Thus the projection matrix to $V^\perp$ is $I - M$.