Math 346: Linear Algebra: Take Home Quiz 5

Solutions:

1. (10 points) Let A and $\mathbf{b}$ be the matrix and vector below:

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & -5 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 12 \\ -8 \\ 32 \end{pmatrix}.$$

   (a) Find the complete solution to $A\mathbf{x} = \mathbf{0}$.

   (b) Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

Solution: *Remark:* You should only need to row reduce once if you use the augmented matrix to $A\mathbf{x} = \mathbf{b}$. We use elimination to reduce to reduced echelon form:

$$(A|\mathbf{b}) = \left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 12 \\ 1 & 2 & 3 & 4 & -8 \\ 1 & 2 & -5 & 0 & 32 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & \frac{5}{2} & 7 \\ 0 & 0 & 1 & \frac{1}{2} & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = (R|\mathbf{c}).$$

Note that R is the reduced echelon form of A.

(a) From the above we know that $(A|\mathbf{0})$ row reduces to

$$(A|\mathbf{0}) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Thus a solution to $A\mathbf{x} = \mathbf{0}$ has the form $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ where

$$x_1 + 2x_2 + \frac{5}{2}x_4 = 0 \quad \text{and} \quad x_3 + \frac{1}{2}x_4 = 0.$$

The pivots are located in the columns for x_1 and x_3 so those are the dependent variables while x_2 and x_4 are free. Solving for the dependent variables we see:

$$x_1 = -2x_2 - \frac{5}{2}x_4 \quad \text{and} \quad x_3 = \frac{-1}{2}x_4.$$

The complete solution is obtained by allowing the free variables x_2 and x_4 to vary in $\mathbb{R}$. Thus,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 - \frac{5}{2}x_4 \\ x_2 \\ \frac{-1}{2}x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{-5}{2} \\ 0 \\ \frac{-1}{2} \\ 1 \end{pmatrix}.$$

Our special solutions are given by

$$\backslash_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \backslash_4 = \begin{pmatrix} \frac{-5}{2} \\ 0 \\ \frac{-1}{2} \\ 1 \end{pmatrix}.$$

We have a complete solution to $A\mathbf{x} = \mathbf{0}$ given by

$$\mathbf{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{-5}{2} \\ 0 \\ \frac{-1}{2} \\ 1 \end{pmatrix} \quad \text{for all } x_2, x_4 \in \mathbb{R}.$$

(b) To find the general solution for $A\mathbf{x} = \mathbf{b}$ observe that our row reduction tells us that solving $A\mathbf{x} = \mathbf{b}$ is the same as solving

$$x_1 + 2x_2 + \frac{5}{2}x_4 = 7 \quad \text{and} \quad x_3 + \frac{1}{2}x_4 = -5.$$

To give the complete solution, we only need to add a particular solution to the solution to $A\mathbf{x} = \mathbf{0}$. We typically find the particular solution by setting the free variables to zero. So set $x_2 = 0$ and $x_4 = 0$. Then our equations become

$$x_1 = 7 \quad \text{and} \quad x_3 = -5.$$

Thus a particular solution is given by

$$\mathbf{x}_p = \begin{pmatrix} 7 \\ 0 \\ -5 \\ 0 \end{pmatrix}.$$

The complete solution is given by $\mathbf{x} = \mathbf{x}_p + x_2\backslash_2 + x_4\backslash_4$ or

$$\mathbf{x} = \begin{pmatrix} 7 \\ 0 \\ -5 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{-5}{2} \\ 0 \\ \frac{-1}{2} \\ 1 \end{pmatrix} \quad \text{for all } x_2, x_4 \in \mathbb{R}.$$

2. Suppose A is an $n \times n$ matrix. Explain why A is invertible if and only if the null space $N(A)$ consists only of the zero vector, i.e. $N(A) = \{\mathbf{0}\}$. Because this is an "if and only if" statement, this amounts to explaining why:
   (a) If A is invertible, then $N(A) = \{\mathbf{0}\}$.

Solution: Suppose A is invertible. Then A has an inverse A^{-1}. We must prove that $N(A) = \{\mathbf{0}\}$. Since $A\mathbf{0} = \mathbf{0}$ we know that $\mathbf{0}$ is in the null space. We need to show that there are no other vectors in the null space. We do this by proving that if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$. So suppose $A\mathbf{v} = \mathbf{0}$. Then by left multiplying by A^{-1} we see

$$\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0},$$

proving $\mathbf{v} = \mathbf{0}$.

(b) If $N(A) = \{\mathbf{0}\}$, then A is invertible.

Hint: Consider the reduced echelon form of A. Recall how elimination relates to row reduction.

Solution: Now suppose that $N(A) = \{\mathbf{0}\}$. We must prove that A is invertible. Following the hint, let R be the reduced echelon form of A. Free variables for the equation $A\mathbf{x} = \mathbf{0}$ come from columns of R without pivots and give special solutions which are non-zero. Thus there must be a pivot in every column. But since the pivots move down and to the right and A is square, the pivots must be in the diagonal entries. Thus $R = I$. Since $A \sim E$, there is an invertible matrix E so that $EA = R = I$. Since E is invertible, we can left multiply by E^{-1} to see that $A = E^{-1}$. Thus A is invertible and its inverse is $A^{-1} = E$.