Math 346: Linear Algebra: Take Home Quiz 4

Solutions:

1. (10 points) Consider the following two vectors in $\mathbb{R}^3$:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Let $S \subset \mathbb{R}^3$ be the *span* of the set of vectors $\{\mathbf{v}, \mathbf{w}\}$ which consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.

(a) Is S a point, line, or plane or a 3-dimensional space?

Solution: It is a plane. The span of $\{\mathbf{v}\}$ is the line through the origin and the point with coordinates given by $\mathbf{v}$. When you add $\mathbf{w}$ it becomes two dimensional (a plane) since $\mathbf{w}$ does not lie on the line through $\mathbf{v}$. The span is the plane containing the origin and the two points given by $\mathbf{v}$ and $\mathbf{w}$.

(b) Is the vector $\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix}$ in S? Why or why not?

Solution: We can frame this question in terms of matrices. Let A be the matrix with two columns given by $\mathbf{v}$ and $\mathbf{w}$. Then all linear combinations of $\mathbf{v}$ and $\mathbf{w}$ are of the form $A\mathbf{z}$ for some $\mathbf{z} \in \mathbb{R}^2$. Thus we are looking to solve the equation $A\mathbf{z} = \mathbf{x}$ for $\mathbf{z}$. We can do this by row elimination applied to the augmented matrix:

$$(A|\mathbf{x}) = \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 1 & 1 & 0 \\ 0 & 2 & -8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -4 \\ 0 & 2 & -8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right).$$

Letting $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, the above matrix corresponds to the equations:

$$z_1 = 4, \quad z_2 = -4, \quad \text{and} \quad 0 = 0.$$

Thus we see that $\mathbf{x}$ is in the span. We have

$$\mathbf{x} = A\begin{pmatrix} 4 \\ -4 \end{pmatrix} \quad \text{or equivalently} \quad \mathbf{x} = 4\mathbf{v} - 4\mathbf{w}.$$

Since $\mathbf{x}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$, we know $\mathbf{x}$ is in S.

(c) Is the vector $\mathbf{y} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$ in S? Why or why not?

Solution: Using the same setup as in the previous part, we see we are trying to solve $A\mathbf{z} = \mathbf{y}$. We have

$$(A|\mathbf{y}) = \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 1 & 1 & 1 \\ 0 & 2 & -9 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 2 & -9 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{array}\right).$$

The last row means that a solution $\mathbf{z}$ to $A\mathbf{z} = \mathbf{y}$ would have to satisfy the equation $0 = -1$ which is impossible. Therefore there is no solution. This means that $\mathbf{z}$ is not a linear combination of $\mathbf{v}$ and $\mathbf{w}$ and that $\mathbf{y}$ is not in S.

2. (10 points) Let $n \geq 2$ and fix an arbitrary $n \times n$ matrix A. We say the $n \times n$ matrix B *commutes* with A if $AB = BA$. Prove that the collection of all $n \times n$ matrices that commute with A is a subspace of the vector space consisting of all $n \times n$ matrices.

Solution: We must show that the set of $n \times n$ matrices that commutes with A is a subspace. For this, we need to show the set is closed under addition and scalar multiplication.

Suppose B_1 and B_2 commute with A (meaning $AB_1 = B_1A$ and $AB_2 = B_2A$) and suppose $c \in \mathbb{R}$ is a scalar. We need to show $B_1 + B_2$ commutes with A and that cB_1 commutes with A. By the distributive law and the fact that B_1 and B_2 commute with A, we have

$$A(B_1 + B_2) = AB_1 + AB_2 = B_1A + B_2A = (B_1 + B_2)A,$$

proving $B_1 + B_2$ commutes with A. Similarly,

$$A(cB_1) = cAB_1 = cB_1A = (cB_1)A,$$

which shows that cB_1 commutes with A.