Math 346: Linear Algebra: Take Home Quiz 3

Solutions:

1. (5 points) What 3×3 matrix A satisfies the following equation?

$$A \begin{pmatrix} 2 & 3 & 4 & 6 \\ 0 & 7 & 3 & -1 \\ -6 & 2 & -3 & -5 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 6 \\ 0 & 7 & 3 & -1 \\ 0 & 11 & 9 & 13 \end{pmatrix}.$$

   Hint: What row operation is being performed?

> **Solution:** Three times the first row is being added to the third row. We perform the same on the 3×3 identity matrix to get A:
>
> $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$
>
> We can check our work my carrying out the multiplication:
>
> $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 6 \\ 0 & 7 & 3 & -1 \\ -6 & 2 & -3 & -5 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 6 \\ 0 & 7 & 3 & -1 \\ 0 & 11 & 9 & 13 \end{pmatrix}.$$

2. Let A and B be $n \times n$ matrices. Expand $(A+B)^3$ so that there are no parenthesis in the result. (Compare homework problems §2.4 # 6, 14.)

> **Solution:**
>
> $$\begin{aligned} (A+B)^3 &= \big[(A+B)(A+B)\big](A+B) \\ &= \big[A(A+B) + B(A+B)\big](A+B) \\ &= [A^2 + AB + BA + B^2](A+B) \\ &= [A^2 + AB + BA + B^2]A + [A^2 + AB + BA + B^2]B \\ &= A^3 + ABA + BA^2 + B^2A + A^2B + AB^2 + BAB + B^3. \end{aligned}$$

3. (10 points) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & x & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where x, y and z are real numbers.

Solution: We preform row reduction on $(A|I)$ reducing it to $(I|A^{-1})$.

First we subtract $x * row4$ from *row3*, subtract $y * row4$ from *row2*, and subtract $z * row4$ from *row1*:

$$\left(\begin{array}{cccc|cccc} 1 & x & y & z & 1 & 0 & 0 & 0 \\ 0 & 1 & x & y & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & x & y & 0 & 1 & 0 & 0 & -z \\ 0 & 1 & x & 0 & 0 & 1 & 0 & -y \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we subtract $x * row3$ from *row2*, and subtract $y * row3$ from *row1*:

$$\left(\begin{array}{cccc|cccc} 1 & x & y & 0 & 1 & 0 & 0 & -z \\ 0 & 1 & x & 0 & 0 & 1 & 0 & -y \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & x & 0 & 0 & 1 & 0 & -y & xy - z \\ 0 & 1 & 0 & 0 & 0 & 1 & -x & x^2 - y \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we subtract $x * row2$ from *row1*:

$$\left(\begin{array}{cccc|cccc} 1 & x & 0 & 0 & 1 & 0 & -y & xy - z \\ 0 & 1 & 0 & 0 & 0 & 1 & -x & x^2 - y \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -x & x^2 - y & 2xy - z - x^3 \\ 0 & 1 & 0 & 0 & 0 & 1 & -x & x^2 - y \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

We see

$$A^{-1} = \begin{pmatrix} 1 & -x & x^2 - y & 2xy - z - x^3 \\ 0 & 1 & -x & x^2 - y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is worth checking your work by verifying $AA^{-1} = I$!