Math 346: Linear Algebra: Take Home Quiz 2

Solutions:

1. (10 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

(a) What are $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

> **Solution:** Since T is linear, for all scalars s and t and all vectors $\mathbf{v}$ and $\mathbf{w}$ we have
>
> $$T(s\mathbf{v} + t\mathbf{w}) = sT(\mathbf{v}) + tT(\mathbf{w}).$$
>
> Observe that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, thus
>
> $$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}T\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 1 \end{pmatrix}.$$
>
> Similarly $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, thus
>
> $$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 4 \end{pmatrix}.$$

(b) Find a matrix A so that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$.

> **Solution:**
>
> $$A = \left[\; T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\right] = \begin{bmatrix} \frac{5}{2} & \frac{1}{2} \\ 1 & 4 \end{bmatrix}.$$

2. (10 points) (a) Suppose C is an invertible matrix and $AC = BC$. Prove that $A = B$.

> **Solution:** Assume $AC = BC$ and that C is invertible. Since C is invertible there is a matrix C^{-1} so that $C^{-1}C = I$ and $CC^{-1} = I$. Since AC and BC are the same matrix, we also get the same matrix when right multiplying further by C^{-1}. Thus $ACC^{-1} = BCC^{-1}$. Simplifying we see $AI = BI$. Since I is the identity matrix $AI = A$ and $BI = B$. Thus $A = B$.

(b) Let $C = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ which is not invertible. Find two 2×2 matrices A and B so that $A \neq B$ but $AC = BC$.

Solution: *Remark:* A solution can be found by experimentation. Just play around! Below I present one approach.

We want $AC = BC$. Observe this is the same as showing $AC - BC$ is the zero matrix. We have $AC - BC = (A - B)C$. So, we are looking for different matrices A and B so that $(A - B)C$ is the zero matrix. Write

$$A - B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then by computation we see

$$(A - B)C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a - b & b - a \\ c - d & d - c \end{pmatrix}.$$

So, $(A - B)C$ is the zero matrix if and only if $a = b$ and $c = d$. So, A and B can be any matrices so that $A - B$ has the form above with $a = b$ and $c = d$. So for example,

$$A = \begin{pmatrix} 7 & 5 \\ 9 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 6 & 4 \\ 4 & -4 \end{pmatrix} \quad \text{work.}$$