Math 346: Practice for the Final (Solutions)

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1. Consider the matrix $A = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

   (a) Find the eigenvalues of A.

   Solution: Solution 1. Since the matrix is upper triangular, the eigenvalues are the diagonal entries. So the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$ (with multiplicity 2).

   Solution 2. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda)^2.$$

   The eigenvalues are the roots which are $\lambda_1 = -1$ and $\lambda_2 = 2$ (with multiplicity 2).

   (b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

   Solution: First we will find an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$. Such an eigenvector must solve $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$. We have

$$A - \lambda_1 I = A + I = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \sim \ldots \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

   So an eigenvector with eigenvalue $\lambda_1 = -1$ is $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

   Now we will find two eigenvectors with eigenvalue $\lambda_2 = 2$. (We must find two since λ_2 has multiplicity 2.) We have

$$A - \lambda_1 I = A - 2I = \begin{pmatrix} -3 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Thus $\mathbf{x}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is an eigenvector with eigenvalue 2 exactly when $-3a + b + 2c = 0$, which we can rewrite as $b = 3a - 2c$. We can take a and c to be our free variables (to avoid fractions) and we get one solution when $a = 1$ and $c = 0$ and one solution when $a = 0$ and $c = 1$. These are the linearly independent eigenvectors

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

   This gives the basis of eigenvectors

$$\left\{ \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

(c) Diagonalize A.

Solution: We have $A = X\Lambda X^{-1}$ where

$$X = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

2. Complete the following definitions:

(a) Let V and W be vector spaces. A transformation (or map) $T : V \to W$ is *linear* if ...

Solution: for every $\mathbf{v}_1, \mathbf{v}_2 \in V$ we have $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ and for every $\mathbf{v} \in V$ and every scalar c we have $T(c\mathbf{v}) = cT(\mathbf{v})$.

(b) Let V be a vector space. A list of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in V$ is called a basis for V if ...

Solution: the set of vectors is linearly independent and spans V.

(c) The dimension of a vector space V is ...

Solution: the number of elements in a basis for V.

3. Let V and W be vector spaces, and let $L : V \to W$ be a linear map. Prove that if the system $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly dependent, then so is the system $L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n)$.

Solution: Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent. Then there are constants $c_1, \ldots, \mathbf{c}_n$ which are not all zero and satisfy

$$c_1\mathbf{v}_1 + \ldots c_n\mathbf{v}_n = \mathbf{0}_V.$$

Since L is linear, we know

$$L(c_1\mathbf{v}_1 + \ldots c_n\mathbf{v}_n) = c_1L(\mathbf{v}_1) + \ldots + c_nL(\mathbf{v}_n).$$

Also we know $L(\mathbf{0}_V) = \mathbf{0}_W$. Thus,

$$c_1L(\mathbf{v}_1) + \ldots + c_nL(\mathbf{v}_n) = \mathbf{0}_W.$$

Since not all of $c_1, \ldots \mathbf{c}_n$ are zero, we know that $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n)\}$ is linearly dependent.

4. Consider the matrix $A = \begin{pmatrix} 0 & 2 & 0 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ -1 & 2 & 0 & 1 \end{pmatrix}$.

Solution: Before we start this problem, it is good to row reduce the matrix. By rearranging rows, we have:

$$A = \begin{pmatrix} 0 & 2 & 0 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ -1 & 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 5 \\ -1 & 2 & 0 & 1 \end{pmatrix}.$$

Now we can clear out the entries under the first pivot by adding a multiple of row 1 to the other rows:

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 5 \\ -1 & 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 5 \end{pmatrix}.$$

Now we can scale row 2 by dividing by two and clear out the entries below the second pivot:

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now we can see the third pivot. We clear out the entry below obtaining:

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The above matrix at right is in echelon form. We continue now to reduced echelon form by clearing out the non-zero-entry above the third pivot:

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for the column space of A.

Solution: The standard solution is to use the pivot columns of the original matrix. Since there are pivots in the first three columns, a basis is given by

$$\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \}.$$

(b) Find a basis for the row space of A.

Solution: The standard solution is to use the non-zero rows of an echelon form of A. We use the reduced echelon form to give the basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(c) Find a basis for the null space of A.

Solution: We will use the standard solution where you get one basis element for each free variable in the equation $A\mathbf{x} = \mathbf{0}$. We already know

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rows above correspond to the equations $x_1+3x_4 = 0$, $x_2+2x_4 = 0$, and $x_3+x_4 = 0$. There is only one free variable, x_4. When $x_4 = 1$ we have $x_1 = -3$, $x_2 = -2$ and $x_3 = -1$. Thus our basis is given by

$$\left\{ \begin{pmatrix} -3 \\ -2 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

5. True or False. You do not need to justify your answers.

(a) If A is a $n \times n$ matrix and I is the $n \times n$ identity matrix, then A and $A + I$ always have the same eigenvectors.

Solution: True. Suppose $\mathbf{x}$ is an eigenvector of A. Then $A\mathbf{x} = \lambda\mathbf{x}$ for some λ. Then

$$(A + I)\mathbf{x} = A\mathbf{x} + I\mathbf{x} = \lambda\mathbf{x} + \mathbf{x} = (\lambda + 1)\mathbf{x}.$$

This shows that $\mathbf{x}$ is an eigenvector of $A+I$ with eigenvalue $\lambda+1$. The same argument works in reverse to show that if $\mathbf{x}$ is an eigenvector of $A+I$ then it is also an eigenvector of A. So, they have the same eigenvalues.

(b) Some matrices have no real eigenvalues.

Solution: True. For example 2×2 rotation matrices do not (except for very special rotation amounts).

(c) If $\mathbf{v}_1, \mathbf{v}_2$ is a linearly dependent system, then one of the vectors must be a scalar multiple of the other.

> **Solution:** True. If they are dependent we can find scalars which are not all zero and satisfy $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. One of the scalars is not zero, so we can solve for the associated vector (yielding either $\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2$ or $\mathbf{v}_2 = -\frac{c_1}{c_2}\mathbf{v}_1$).

(d) The equation $A\mathbf{x} = \mathbf{0}$ can have no solution.

> **Solution:** False. There is always the trivial solution $\mathbf{x} = 0$.

(e) If V and W are vector spaces, then so is the collection of all linear maps from V to W.

> **Solution:** True. We discussed that you can add and scale linear maps. If $T_1 : V \to W$ and $T_2 : V \to W$ are linear maps then so is $T_1 + T_2$ which sends $\mathbf{v} \in V$ to $T_1(\mathbf{v}) + T_2(\mathbf{v})$. If $T : V \to W$ is linear and c is a scalar then the map cT which sends $\mathbf{v} \in V$ to $cT(\mathbf{v})$ is also linear.

(f) If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of a square matrix A, then $\mathbf{v} + \mathbf{w}$ is always either the zero vector or an eigenvector of A.

> **Solution:** False. For a counterexample consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. The vectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors but their non-zero sum $\mathbf{x}_1 + \mathbf{x}_2$ is not.

(g) If W_1 and W_2 are two subspaces of a vector space V, then the set of all vectors of the form $\mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ is always also a subspace of V.

> **Solution:** True. You can see that the set of vectors of the form $\mathbf{w}_1 + \mathbf{w}_2$ is closed under addition and scalary multiplication.

(h) Some matrices have a right inverse, but have no left inverse.

> **Solution:** True. An example is
>
> $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
>
> which has a right-inverse given by A^T. The matrix does not have a left inverse because its rank is 2, and the product of two matrices can not have rank larger than the rank of either of the matrices in the product.

6. The matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ can be written as QDQ^{-1}, where:

$$Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad Q^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

(a) List all the eigenvalues of A, and list one eigenvector for each eigenvalue.

Solution: The matrix has an eigenvalue of $\lambda_1 = 2$ with an associated eigenvector $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The matrix has an eigenvalue of $\lambda_2 = 3$ with an associated eigenvector $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. These are all the eigenvalues.

(b) Give a diagonalization of A^{-1}. You can leave Q and Q^{-1} in your solution but not D.

Solution: If $A = QDQ^{-1}$ then $A^{-1} = QD^{-1}Q^{-1}$ and we have

$$D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

(c) Give four different square roots of A expressed in terms of Q. (A *square root* of A is a matrix B so that $B^2 = A$.)

Solution: Four different square roots of A are:

$$Q\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}, \quad Q\begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}, \quad Q\begin{pmatrix} -\sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}, \quad \text{and} \quad Q\begin{pmatrix} -\sqrt{2} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}.$$

7. Give short answers to the following questions. No explanations are necessary.

(a) What are all the eigenvectors of the 5×5 identity matrix?

Solution: All non-zero vectors in $\mathbb{R}^5$.

(b) What is the area of the parallelogram in the plane with vertices at the points $(0,0)$, $(2,3)$, $(4,8)$ and $(2+4, 3+8)$?

Solution: The absolute value of the determinant

$$\begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix} = 16 - 12 = 4.$$

(c) An $n \times m$ matrix has rank r. What is the dimension of the null space? What is the dimension of the row space?

Solution: Let A be the $n \times m$ matrix. Then the dimension of the null space is $\dim N(A) = m - r$ and the dimension of the row space is $\dim C(A^T) = r$.

(d) If A is 3×3 and $\det A = -5$, then what is $\det(2A^2)$?

Solution: Doubling a matrix causes all its rows to scale by 2 which causes a 3×3 determinant to multiply by eight. Squaring a matrix causes its determinant to square. Thus

$$\det(2A^2) = 8\det(A)^2 = 8(-5)^2 = 8 \cdot 25 = 200.$$

(e) If V is a vector space of dimension 7, what are the possible dimensions of a subspace of V?

Solution: 0, 1, 2, 3, 4, 5, 6 or 7.

8. Let A be a matrix. Suppose that the homogeneous equation $A\mathbf{x} = 0$ has a non-trivial solution $\mathbf{x}_h$. Prove that the equation $A\mathbf{x} = \mathbf{b}$ has either zero or infinitely many solutions.

Solution: Suppose that $A\mathbf{x} = \mathbf{b}$ does not have zero solutions. We must show it has infinitely many solutions. Since it does not have zero solutions, $A\mathbf{x} = \mathbf{b}$ has at least one solution, call it $\mathbf{x}_s$. We claim that $\mathbf{x}_s + c\mathbf{x}_h$ is a solution for each scalar c. To see this observe that

$$A(\mathbf{x}_s + c\mathbf{x}_h) = A(\mathbf{x}_s) + cA(\mathbf{x}_h) = \mathbf{b} + c\mathbf{0} = \mathbf{b}.$$

Since there are infinitely many scalars, we have found infinitely many solutions to $A\mathbf{x} = \mathbf{b}$.

9. Prove that if B is a left inverse of A, and C is a right inverse of A, then $B = C$.

Solution: We are given that $BA = I$ and $AC = I$. Now take the equation $BA = I$ and right multiply by C. We have $(BA)C = IC$. By associativity and because I is the identity matrix we know this is the same as $B(AC) = C$. We also know $AC = I$ so this can be rewritten as $BI = C$. Again since I is the identity matrix we see $B = C$.

10. Let A be an $n \times n$ matrix. Recall that a square matrix is invertible if and only if its determinant is non-zero.

(a) Complete the following definition: The *characteristic polynomial* of A is ...

Solution: the polynomial in λ given by $\det(A - \lambda I)$.

(b) Show that if λ_0 is an eigenvalue, then it is a root of the characteristic polynomial. (*Hint:* Why isn't $A - \lambda_0 I$ invertible?)

Solution: If λ_0 is an eigenvalue then there is a non-zero vector $\mathbf{x}_0$ so that $A\mathbf{x}_0 = \lambda_0\mathbf{x}_0$. This means that $(A - \lambda_0 I)\mathbf{x}_0 = \mathbf{0}$. But since $(A - \lambda_0 I)\mathbf{0}$ is also equal to zero, the matrix can not be invertible (the associated linear map is not one-to-one). Since $A - \lambda_0 I$ is not invertible we have $\det(A - \lambda_0 I) = 0$. That is, λ_0 is a root of the characteristic polynomial.

(c) Show that if λ_0 is a root of the characteristic polynomial, then λ_0 is an eigenvalue. (*Hint:* How many solutions do you have for the eigenvector equation?)

Solution: Suppose λ_0 is a root of the characteristic polynomial. Then $\det(A-\lambda_0 I) = 0$. This means that $A - \lambda_0 I$ is not invertible, which means there is not a pivot in every column or equivalently there is a column without a pivot. Columns without pivots correspond to free variables which give a non-zero solution $\mathbf{x}_0$ to the equation $(A - \lambda_0 I)x = \mathbf{0}$. Since $(A - \lambda_0 I)x_0 = \mathbf{0}$ we see that $A\mathbf{x}_0 = \lambda_0 \mathbf{x}_0$. Since $\mathbf{x}_0$ is not zero, we see λ_0 is an eigenvalue of A by definition.

11. Find the determinant of the following matrix using whatever method you like.

$$A = \begin{pmatrix} 2 & 2 & -6 & 9 \\ 0 & 0 & 6 & 1 \\ 0 & 2 & 3 & 3 \\ 1 & 2 & -3 & 4 \end{pmatrix}.$$

Solution: Using the row reduction method:

$$\begin{vmatrix} 2 & 2 & -6 & 9 \\ 0 & 0 & 6 & 1 \\ 0 & 2 & 3 & 3 \\ 1 & 2 & -3 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 2 & 3 & 3 \\ 2 & 2 & -6 & 9 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & -2 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 3 & 4 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 3 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -7 \end{vmatrix} = -(1)(2)(3)(-7) = 42.$$

Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$. Also consider the following matrices:

$$B = \begin{pmatrix} a_{1,1} & -2a_{1,2} & a_{1,3} \\ -2a_{2,1} & 4a_{2,2} & -2a_{2,3} \\ a_{3,1} & -2a_{3,2} & a_{3,3} \end{pmatrix}, \quad C = \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{1,1} & a_{1,2} & a_{1,3} \end{pmatrix}, \quad D = \begin{pmatrix} a_{1,1} & 0 & 0 & a_{1,2} & a_{1,3} \\ a_{2,1} & 0 & 0 & a_{2,2} & a_{2,3} \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ a_{3,1} & 0 & 0 & a_{3,2} & a_{3,3} \end{pmatrix}.$$

(a) Find an equation relating $\det B$ to $\det A$. Briefly justify your answer.

Solution: The matrix B is obtained from A by scaling the second row and the second column by (-2). Therefore,

$$\det B = (-2)(-2)\det A = 4\det A.$$

(b) Find an equation relating $\det C$ to $\det A$. Briefly justify your answer.

Solution: The matrix C is obtained from A by switching the first two rows and then switching the last two rows. Each of these changes results in a sign change. So,

$$\det C = (-1)(-1)\det A = \det A.$$

(c) Find an equation relating $\det D$ to $\det A$. Briefly justify your answer.

Solution: Using cofactor expansion twice in the second column, we see that:

$$\det D = -3 \begin{vmatrix} a_{1,1} & 0 & a_{1,2} & a_{1,3} \\ a_{2,1} & 0 & a_{2,2} & a_{2,3} \\ 0 & 5 & 0 & 0 \\ a_{3,1} & 0 & a_{3,2} & a_{3,3} \end{vmatrix} = (-3)(-5) \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = 15 \det A.$$

12. (a) Complete the following definition: The null space of an $m \times n$ matrix A is ...

Solution: the set of all vectors $\mathbf{v} \in \mathbb{R}^n$ so that $A\mathbf{v} = \mathbf{0}$.

(b) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that

$$\dim N(AB) \geq \dim N(B).$$

(*Hint:* What is the relationship between $N(AB)$ and $N(B)$?)

Solution: Observe that any vector in $N(B)$ is also in $N(AB)$. This is because if $B\mathbf{v} = \mathbf{0}$, then $AB\mathbf{v} = A\mathbf{0} = \mathbf{0}$. So, $N(AB)$ contains $N(B)$. Therefore, $\dim N(AB) \geq \dim N(B)$.

More details: The logic here can be reduced to bases. If $N(B)$ is dimension d, then $N(B)$ has a basis consisting of d-vectors. These d linearly independent vectors also lie in $N(AB)$. A linearly independent collection of vectors in $N(AB)$ can not have more than $\dim N(AB)$ vectors in it. (See Chapter 2, Proposition 5.2.). Therefore, $\dim N(AB) \geq d$.

13. Let $A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$.

(a) Find a basis for the null space of A.

Solution: We can row reduce to reduced echelon form:

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -2 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

We have free variables in columns 3 and 4 giving us the basis for $N(A)$:

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\}.$$

(b) Find an orthonormal basis for the null space of A.

Solution: Well this was too simple! The vectors are already orthogonal, so we just need to make them unit vectors. An orthonormal basis is

$$\{\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\}.$$

(c) Give a matrix P so that $P\mathbf{x}$ is the projection of $\mathbf{x}$ onto $N(A)$.

Solution: Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the two vectors forming the orthonormal basis for $N(A)$ found in the previous part. Then $P = \mathbf{x}_1\mathbf{x}_1^T + \mathbf{x}_2\mathbf{x}_2^T$ (i.e., P is the sum of the projections onto the one-dimensional subspaces associated to the basis). We have

$$\mathbf{x}_1\mathbf{x}_1^T = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2\mathbf{x}_2^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This makes P equal to the sum,

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

14. Show that the map $L : \mathbb{R} \to \mathbb{R}$ given by $L(t) = 3t - 1$ is not a linear function.

Solution: If L were linear then $L(cx) = cL(x)$ for any scalar c and any x in the domain. But for $x = 1$ and $c = 0$ we have

$$L(0 \cdot 1) = L(0) = -1 \quad \text{and} \quad 0 \cdot L(1) = 0 \cdot (-1) = 0.$$

15. Given that A and B are invertible $n \times n$ matrices, prove that the product AB is invertible.

Solution: Since A and B are invertible, there inverses A^{-1} and B^{-1} exist. Let $C = B^{-1}A^{-1}$. We claim that C is the inverse of AB. Observe:

$$(AB)C = (AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$C(AB) = B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$