Math 346: Practice for the Final

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. New material covered by the final includes §4.2 and 4.4, §5.1-5.3, §6.1, 6.2, and 6.4, as well as §9.1. Earlier in the semester we covered §1.1-1.3, §2.1-2.6, §3.1-3.5, §4.1 and §8.1 and some material covered in handouts (on linear transformations and function and matrix inverses). The most recent material will be emphasized (taking up approximately half the final), but older material will also be tested. You are expected to know all material covered in the course up until now.

Basic concepts. You should understand and be able to work with basic terms used in the study of Linear Algebra. You should also be able to define most of these terms which are discussed by the book and also were discussed in class. Newer concepts:

projection (p. 206), *projection matrix* (p. 206), *orthogonal subspaces* (p. 207), *orthogonal compliements* (p. 207), *orthogonal basis* (p. 233), *orthonormal basis* (p. 233), *orthogonal matrix* (p. 234), *The Graham-Schmidt Process (for creating an orthonormal basis)* (p. 237), *Determinant* (p. 247), *cofactor* (p. 263, 267), *cofactor expansion/formula* (p. 263), *Cramer's Rule* (p. 274), *connection between determinant and volume* (p. 273), *eigenvectors* (p. 289), *eigenvalues* (p. 292), *characteristic polynomial* (p. 292), *diagonalization of a matrix* (p. 305), *similar matrix* (p. 308), *The Spectral Theorem* (p. 339), *complex number* (p. 431), *real part* (p. 431), *imaginary part* (p. 431), *complex conjugation* (p. 431), *absolute value of a complex number* (p. 433), *argument (or angle) of a complex number* (p. 433), *formula for a power of a complex number* (p. 434),

Older concepts covered:

linear combination (p. 1-3), dot product of vectors (p. 11), length of a vector (p. 12), unit vector (p. 13, 14), matrix-vector multiplication (p. 22, 36-37), coefficient matrix (p. 33, 36), back substitution (p. 34), elimination or row reduction (p. 46), elementary or elimination matrix (p. 60), matrix multiplication (p. 61), augmented matrix (p. 63), inverse matrix (p.83, handout 2), upper triangular matrix (p. 46, 97), lower triangular matrix (p. 98), triangular matrix (p. 52, 89), LU-factorization (p. 97), Linear transformation (p. 401, handout 1), composition (handout 1), identity transformation (p. 402), matrix of a linear transformation (handout 1), identity matrix (p. 37, handout 1), *vector space* (p. 123), *subspace* (p. 125), *column space* (p. 127), *span* (p. 128 & 167), *nullspace* (p. 135), *special solution* (p. 135), *reduced row echelon form* (p. 137), *rank* (p. 139), *complete solution* (p. 153), *linearly independent* (p. 165), *row space* (p. 168), *basis* (p. 168), *standard basis* (p. 169), *dimension of a space* (p. 171), *left null space* (p. 181), *Fundamental Theorem of Linear Algebra, Part 1* (p. 185) *orthogonal subspaces* (p. 195) *orthogonal complement* (p. 195) *Fundamental Theorem of Linear Algebra, Part 1* (p. 198)

Techniques. Knowing the new material we have covered, you should be able to:

- Compute the projection matrix associated to a subspace of $\mathbb{R}^n$.
- Find othogonal or orthonormal bases for subspaces of $\mathbb{R}^n$.
- Compute the determinant of an $n \times n$ matrix (using cofactor expansion or row reduction).

- Understand the meaning of the determinant both in terms of volumes of parallelepipeds and change in volume caused by applying a linear transformation.
- Compute eigenvalues and eigenvectors of a matrix. Diagonalize a matrix. Use these objects to understand a transformation (for example, the powers of a matrix.)
- Compute with complex numbers especially when they appear in eigenvector computations. You should be able to find all complex roots of a real number, and use complex numbers arising from the quadratic formula.
- State and use the spectral theorem on diagonalization of symmetric matrices.

From prior work in the class, you should be able to:

- Apply elimination to a matrix, reducing it to upper triangular form or the identity matrix. Recognize pivots in the reduced matrix and understand their meaning.
- Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ in its various forms (system of linear equations, matrix equation, solution to linear combination problem).
- Demonstrate an understanding of linear transformations. You should be able to use the definition of *linear transformation* and manipulate linear transformations. Find the matrix of a linear transformation.
- Decide if a matrix is invertible and find its inverse if it is invertible. Write an invertible matrix as a product of elementary (or elimination) matrices .
- Factor a matrix A into the form LU, where L is a lower triangular matrix and U is an upper triangular matrix. You should be able to use this factorization to solve the equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.
- Prove that a subset of a vector space is a subspace.
- Reduce a matrix to reduced row echelon form.
- Find complete solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ and make use of the relationships between these equations.
- Find the rank of a matrix. Understand how it relates to the dimensions of the four fundamental subspaces: the column space, null space, row space, and left null space of a matrix.
- Find bases for the four fundamental subspaces. Find bases for subspaces given in other ways (such as by a span or zero sets to linear equations).
- Demonstrate that two subspaces are orthogonal. Compute the orthogonal complement of a subspace.

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test, and they do not cover all possible problems I would ask on a test. You should also make sure you know how to do all homework problems that were assigned! Note the problems concentrate on recent material, but earlier material will be covered as well. Please see the other reviews for problems testing earlier material.

1. Consider the matrix $A = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.
   (a) Find the eigenvalues of A.
   (b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.
   (c) Diagonalize A.

2. Complete the following definitions:
   (a) Let V and W be vector spaces. A transformation (or map) $T : V \to W$ is *linear* if ...
   (b) Let V be a vector space. A list of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in V$ is called a basis for V if ...
   (c) The dimension of a vector space V is ...

3. Let V and W be vector spaces, and let $L : V \to W$ be a linear map. Prove that if the system $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly dependent, then so is the system $L(\mathbf{v}_1), \ldots, L(\mathbf{v}_n)$.

4. Consider the matrix $A = \begin{pmatrix} 0 & 2 & 0 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ -1 & 2 & 0 & 1 \end{pmatrix}$.
   (a) Find a basis for the column space of A.
   (b) Find a basis for the row space of A.
   (c) Find a basis for the null space of A.

5. True or False. You do not need to justify your answers.
   (a) If A is a $n \times n$ matrix and I is the $n \times n$ identity matrix, then A and $A + I$ always have the same eigenvectors.
   (b) Some matrices have no real eigenvalues.
   (c) If $\mathbf{v}_1, \mathbf{v}_2$ is a linearly dependent system, then one of the vectors must be a scalar multiple of the other.
   (d) The equation $A\mathbf{x} = \mathbf{0}$ can have no solution.
   (e) If V and W are vector spaces, then so is the collection of all linear maps from V to W.
   (f) If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of a square matrix A, then $\mathbf{v} + \mathbf{w}$ is always either the zero vector or an eigenvector of A.
   (g) If W_1 and W_2 are two subspaces of a vector space V, then the set of all vectors of the form $\mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1 \in W_1$ an $\mathbf{w}_2 \in W_2$ is always also a subspace of V.
   (h) Some matrices have a right inverse, but have no left inverse.

6. The matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ can be written as QDQ^{-1}, where:

$$Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad Q^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

   (a) List all the eigenvalues of A, and list one eigenvector for each eigenvalue.

   (b) Give a diagonalization of A^{-1}. You can leave Q and Q^{-1} in your solution but not D.

   (c) Give four different square roots of A expressed in terms of Q. (A *square root* of A is a matrix B so that $B^2 = A$.)

7. Give short answers to the following questions. No explanations are necessary.

   (a) What are all the eigenvectors of the 5×5 identity matrix?

   (b) What is the area of the parallelogram in the plane with vertices at the points $(0, 0)$, $(2, 3)$, $(4, 8)$ and $(2 + 4, 3 + 8)$?

   (c) An $n \times m$ matrix has rank r. What is the dimension of the null space? What is the dimension of the row space?

   (d) If A is 3×3 and $\det A = -5$, then what is $\det(2A^2)$?

   (e) If V is a vector space of dimension 7, what are the possible dimensions of a subspace of V?

8. Let A be a matrix. Suppose that the homogeneous equation $A\mathbf{x} = 0$ has a non-trivial solution $\mathbf{x}_h$. Prove that the equation $A\mathbf{x} = \mathbf{b}$ has either zero or infinitely many solutions.

9. Prove that if B is a left inverse of A, and C is a right inverse of A, then $B = C$.

10. Let A be an $n \times n$ matrix. Recall that a square matrix is invertible if and only if its determinant is non-zero.

    (a) Complete the following definition: The *characteristic polynomial* of A is ...

    (b) Show that if λ_0 is an eigenvalue, then it is a root of the characteristic polynomial. (*Hint:* Why isn't $A - \lambda_0 I$ invertible?)

    (c) Show that if λ_0 is a root of the characteristic polynomial, then λ_0 is an eigenvalue. (*Hint:* How many solutions do you have for the eigenvector equation?)

11. Find the determinant of the following matrix using whatever method you like.

$$A = \begin{pmatrix} 2 & 2 & -6 & 9 \\ 0 & 0 & 6 & 1 \\ 0 & 2 & 3 & 3 \\ 1 & 2 & -3 & 4 \end{pmatrix}.$$

Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$. Also consider the following matrices:

$$B = \begin{pmatrix} a_{1,1} & -2a_{1,2} & a_{1,3} \\ -2a_{2,1} & 4a_{2,2} & -2a_{2,3} \\ a_{3,1} & -2a_{3,2} & a_{3,3} \end{pmatrix}, \quad C = \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{1,1} & a_{1,2} & a_{1,3} \end{pmatrix}, \quad D = \begin{pmatrix} a_{1,1} & 0 & 0 & a_{1,2} & a_{1,3} \\ a_{2,1} & 0 & 0 & a_{2,2} & a_{2,3} \\ 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ a_{3,1} & 0 & 0 & a_{3,2} & a_{3,3} \end{pmatrix}.$$

(a) Find an equation relating $\det B$ to $\det A$. Briefly justify your answer.

(b) Find an equation relating $\det C$ to $\det A$. Briefly justify your answer.

(c) Find an equation relating $\det D$ to $\det A$. Briefly justify your answer.

12. (a) Complete the following definition: The null space of an $m \times n$ matrix A is ...

    (b) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that

$$\dim N(AB) \geq \dim N(B).$$

    (*Hint:* What is the relationship between $N(AB)$ and $N(B)$?)

13. Let $A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$.

    (a) Find a basis for the null space of A.

    (b) Find an orthonormal basis for the null space of A.

    (c) Give a matrix P so that $P\mathbf{x}$ is the projection of $\mathbf{x}$ onto $N(A)$.

14. Show that the map $L : \mathbb{R} \to \mathbb{R}$ given by $L(t) = 3t - 1$ is not a linear function.

15. Given that A and B are invertible $n \times n$ matrices, prove that the product AB is invertible.