Math 346: Practice for Midterm 2

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §3.1-3.5 and §4.1. Knowledge of earlier material will also be necessary to do well on the test, but earlier material will not be explicitly tested. You are expected to know all material covered in the course up until now.

Basic concepts. You should understand and be able to work with basic terms used in the study of Linear Algebra. You should also be able to define most of these terms which are discussed by the book and also were discussed in class.

vector space (p. 123), *subspace* (p. 125), *column space* (p. 127), *span* (p. 128 & 167), *nullspace* (p. 135), *special solution* (p. 135), *reduced row echelon form* (p. 137), *rank* (p. 139), *complete solution* (p. 153), *linearly independent* (p. 165), *row space* (p. 168), *basis* (p. 168), *standard basis* (p. 169), *dimension of a space* (p. 171), *left null space* (p. 181), *Fundamental Theorem of Linear Algebra, Part 1* (p. 185) *orthogonal subspaces* (p. 195) *orthogonal complement* (p. 195) *Fundamental Theorem of Linear Algebra, Part 1* (p. 198)

Techniques. You should be able to:

- Prove that a subset of a vector space is a subspace.
- Reduce a matrix to reduced row echelon form.
- Find complete solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ and make use of the relationships between these equations.
- Find the rank of a matrix. Understand how it relates to the dimensions of the four fundamental subspaces: the column space, null space, row space, and left null space of a matrix.
- Find bases for the four fundamental subspaces. Find bases for subspaces given in other ways (such as by a span or zero sets to linear equations).
- Demonstrate that two subspaces are orthogonal. Compute the orthogonal complement of a subspace.

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test, and they do not cover all possible problems I would ask on a test. You should also make sure you know how to do all homework problems that were assigned!

1. Suppose V is a vector space.
   (a) Complete the following definition:
   A set vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is called *linearly independent* if ...

Solution: the only list of scalars satisfying

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

are given by $c_1 = c_2 = \ldots = c_p = 0$.

(b) Suppose the list of three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent. Prove that the system of two vectors $\mathbf{v}_1, \mathbf{v}_2$ is linearly independent.

Solution: Assume that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Then the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{1}$$

is when $c_1 = c_2 = c_3 = 0$. Now we will prove that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Suppose d_1 and d_2 are scalars satisfying

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 = \mathbf{0}.$$

We must prove that $d_1 = d_2 = 0$. Observe that with d_1 and d_2 as above, we have

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{0}.$$

Then since the only solution to equation (1) is $c_1 = c_2 = c_3 = 0$ we must have $d_1 = 0$ and $d_2 = 0$. This proves that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

2. Let V be a vector space.

(a) Complete the following definition:
A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in V spans V if ...

Solution: for every $\mathbf{w} \in V$ there are scalars $c_1, \ldots, c_p \in \mathbb{R}$ so that

$$\mathbf{w} = c_1\mathbf{v}_1 + \ldots + \mathbf{c}_p\mathbf{v}_p.$$

(b) Suppose that the set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\} \subset V$ is linear independent but does not span V. Fill in the blanks to complete the proof that you can find another vector $\mathbf{v}_3 \in V$ so that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent.

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2$ be a system which is linear independent but does not span V. Since $\mathbf{v}_1, \mathbf{v}_2$ does not generate, there is a vector $\mathbf{v}_3$ such that

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We claim that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent. Suppose to the contrary that this system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linear dependent. Then, there would be scalars $c_1, c, c_3 \in \mathbb{R}$ which are not all zero and satisfy

We will prove that this causes a contradiction. First if $c_3 = 0$, then we have a contradiction because

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Second if $c_3 \neq 0$, then we have a contradiction because

Since one of these two possibilities must occur, it can not be that the system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent. Therefore, we have shown that this system is linearly independent.

Solution: Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2$ be a system which is linear independent but does not span V. Since $\mathbf{v}_1, \mathbf{v}_2$ does not span, there is a vector $\mathbf{v}_3$ such that

$$\underline{\mathbf{v}_3 \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \quad \text{for any } c_1, c_2 \in \mathbb{R}}.$$

We claim that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent. Suppose to the contrary that this system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linear dependent. Then, there would be scalars $c_1, c_2, c_3 \in \mathbb{R}$ which are not all zero and satisfy

$$\underline{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}}.$$

We will prove that this causes a contradiction. First if $c_3 = 0$, then we have a contradiction because
then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ with $c_1 \neq 0$ or $c_2 \neq 0$ which is impossible since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Second if $c_3 \neq 0$, then we have a contradiction because <u>we can solve for $\mathbf{v}_3$ and see</u>

$$\underline{\mathbf{v}_3 = -\frac{c_1}{c_3}\mathbf{v}_1 - \frac{c_2}{c_3}\mathbf{v}_2,}$$

<u>which shows that $\mathbf{v}_3$ was in fact in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$</u>. Since one of these two possibilities must occur, it can not be that the system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent. Therefore, we have shown that this system is linearly independent.

3. Let A be an $m \times n$ matrix. Complete the following definitions.

   (a) The *column space* of A is ...

Solution: the collection of all linear combinations of the columns vectors of A. Variants of this definition are fine. Also okay:

- the span of the columns of A.
- the collection of vectors of the form $A\mathbf{x}$.

- the set of $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Recall that the *null space* $N(A)$ is the set of vectors $\mathbf{v} \in \mathbb{R}^n$ so that $A\mathbf{v} = \mathbf{0}$. Prove that $N(A)$ is a subspace of $\mathbb{R}^n$.

Solution: To prove $N(A)$ is a subspace of $\mathbb{R}^n$ we need to show that $\mathbf{0} \in N(A)$ and that $N(A)$ is closed under addition and scalar multiplication.

To see $\mathbf{0} \in N(A)$ observe that $A\mathbf{0} = \mathbf{0}$.

To see that $N(A)$ is closed under addition suppose that $\mathbf{x}_1, \mathbf{x}_2 \in N(A)$. We need to show that $\mathbf{x}_1 + \mathbf{x}_2 \in N(A)$. Since $\mathbf{x}_1, \mathbf{x}_2 \in N(A)$ we know $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. Using the distributive property for matrix multiplication we see

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{x}_1 + \mathbf{x}_2 \in N(A)$.

To see that $N(A)$ is closed under scalar multiplication suppose that $\mathbf{x} \in N(A)$ and $c \in \mathbb{R}$. We need to show that $c\mathbf{x} \in N(A)$. Because we can pull scalars out of matrix products we have

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c\mathbf{0} = \mathbf{0},$$

so $c\mathbf{x} \in N(A)$.

4. True or False.

(a) If A is a square matrix with linearly independent columns, then A is invertible.

Solution: True. Let A be the square matrix with linearly independent columns. Then there is a pivot in every row. Since pivots occur in different rows and columns, there is also also pivot in every column. So, the reduced echelon form of A is I. So there is an invertible matrix E with $EA = I$. Then $A = E^{-1}$ and A is invertible with inverse E.

(b) Five vectors in $\mathbb{R}^4$ can span $\mathbb{R}^4$.

Solution: True. For example $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ and $\mathbf{0}$ span $\mathbb{R}^4$. (Here $\mathbf{e}_i$ are the standard basis vectors for $\mathbb{R}^4$; the columns of the identity matrix.)

(c) You can find five linearly independent vectors in $\mathbb{R}^4$.

Solution: False. This would mean $\mathbb{R}^4$ contained a 5-dimensional subspace, but we know $\mathbb{R}^4$ is 4-dimensional.

(d) If the subspaces S and T of $\mathbb{R}^n$ share a common non-zero vector $\mathbf{v}$, then S and T are not orthogonal.

Solution: True. For S and T to be orthogonal any $\setminus \in S$ and any $\mathbf{t} \in T$ must be orthogonal. If $\mathbf{v}$ lies in both spaces then $\mathbf{v}$ must be orthogonal to itself. But this impossible in $\mathbb{R}^n$ since

$$\mathbf{v} \cdot \mathbf{v} = \sum_{i=1}^{n} v_i^2.$$

Thus $\mathbf{v} \cdot \mathbf{v} > 0$ unless all entries of $\mathbf{v}$ are zero.

(e) Two planar subspaces of $\mathbb{R}^3$ can be orthogonal.

Solution: No because of the previous part. Two planes intersect in a line, which contains non-zero vectors. Since the two subspaces contain a common non-zero vector, they can not be orthogonal.

5. Find a basis for the column space of

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}.$$

What is the rank of A? How about the dimension of the null space $N(A)$?

Solution: We begin by reducing A to its reduced echelon form. Omitting the work we see:

$$A \sim R = \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now recall that a basis for the column space is given by the pivot columns of A. There is a pivot in the first and second column, so a basis is

$$\left\{ \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ 8 \end{pmatrix} \right\}.$$

The rank of A is 2 because it has 2 pivots. The dimension of the null space of A is the number of columns minus the rank, which is $4 - 2 = 2$.

6. Find a basis for the plane of solutions to the equation $x + 2y + 3z = 0$.

Solution: We need to convert this to something we know how to solve. Observe that

$$x + 2y + 3z = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

So we are looking for a basis to $N(A)$ where $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. A basis is given by the special solutions, one for each free variable. Since A is already in reduced echelon form, we can see the x-variable is dependent while the y- and z-variables are free. Solving for the free variables yields $x = -2y - 3z$. So our solutions have the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y - 3z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2y \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

So a basis for $N(A)$ is given by $\{ \begin{pmatrix} -2y \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \}$.

7. Suppose A is a 4×5 matrix and the reduced row echelon form of A is given by

$$R = \begin{pmatrix} 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for $N(A)$.

Solution: A basis for $N(A)$ is given by the special solutions to $A\mathbf{x} = \mathbf{0}$. Since we are given the reduced echelon form, we can see the dependent variables are x_1, x_3 and x_5 and the free variables are x_2 and x_4. The solution set for $R\mathbf{x} = \mathbf{0}$ coincides with the solution set to $A\mathbf{x} = \mathbf{0}$ and converting $R\mathbf{x} = \mathbf{0}$ to equations yields:

$$x_1 - x_2 - 2x_4 = 0, \quad x_3 + 3x_4 = 0 \quad \text{and} x_5 = 0.$$

Solving for the dependent variables yields:

$$x_1 = x_2 + 2x_4 = 0, \quad x_3 = -3x_4 \quad \text{and} x_5 = 0.$$

So an arbitrary solution has the form

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 + 2x_4 \\ x_2 \\ -3x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}.$$

So, a basis for $N(A)$ is given by $\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \}$.

(b) Suppose $A\mathbf{x}_p = \mathbf{b}$ where

$$\mathbf{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

Solution: Solutions to $A\mathbf{x} = \mathbf{b}$ have the form $\mathbf{x}_p + \mathbf{s}$ where $\mathbf{s}$ is a solution to $A\mathbf{x} = \mathbf{0}$. A complete solution is then

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}.$$

8. Suppose the matrix A is 9×7 and has rank 4. What are the dimensions of $C(A)$, $C(A^T)$, $N(A)$ and $N(A^T)$?

Solution: The number of rows of A is $m = 9$, the number of columns is $n = 7$ and the rank is $r = 4$. We have $\dim C(A) = \dim C(A^T) = r = 4$, $\dim N(A) = n - r = 7 - 4 = 3$ and $\dim N(A^T) = m - r = 9 - 4 = 5$. (These formulas can be memorized!)

9. Let A be an $m \times n$ matrix. Prove that the row space and null spaces of A are orthogonal.

Solution: Solution 1: We need to prove that $\mathbf{y} \cdot \mathbf{x} = 0$ for any element $\mathbf{y}$ of the row space and any element $\mathbf{x}$ of the null space of A. Recall $\mathbf{y} \cdot \mathbf{x} = \mathbf{y}^T\mathbf{x}$. Also if y is in the row space $C(A^T)$, we have $\mathbf{y} = A^T\mathbf{z}$ for some vector $\mathbf{z}$. Then $\mathbf{y}^T = \mathbf{z}^T A$. Since $\mathbf{x} \in N(A)$ then $A\mathbf{x} = \mathbf{0}$. Then

$$\mathbf{y} \cdot \mathbf{x} = \mathbf{y}^T\mathbf{x} = (\mathbf{z}^T A)\mathbf{x} = \mathbf{z}^T(A\mathbf{x}) = \mathbf{z}^T\mathbf{0} = 0.$$

Solution 2: We need to prove that $\mathbf{y} \cdot \mathbf{x} = 0$ for any element $\mathbf{y}$ of the row space and any element $\mathbf{x}$ of the null space of A. Fix $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{r}_i$ be the rows vectors of A. Then $A = \begin{pmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_m^T \end{pmatrix}$. Since $\mathbf{x} \in N(A)$,

$$\mathbf{0} = A\mathbf{x} = \begin{pmatrix} \mathbf{r}_1^T\mathbf{x} \\ \vdots \\ \mathbf{r}_m^T\mathbf{x} \end{pmatrix}.$$

Thus, the dot product $\mathbf{r}_i^T\mathbf{x} = 0$ for each i. Since $\mathbf{y}$ is in the row space, we have

$$\mathbf{y} = c_1\mathbf{r}_1 + \ldots + c_m\mathbf{r}_m$$

for some scalars $c_1, \ldots, c_m \in \mathbb{R}$. Then by the distributive property of matrix multiplication,

$$\mathbf{y} \cdot \mathbf{x} = \mathbf{y}^T\mathbf{x} = (c_1\mathbf{r}_1^T + \ldots + c_m\mathbf{r}_m^T)\mathbf{x} = c_1(\mathbf{r}_1^T\mathbf{x}) + \ldots + c_m(\mathbf{r}_m^T\mathbf{x}) = c_1\mathbf{0} + \ldots + c_m\mathbf{0} = \mathbf{0}.$$

10. Fix positive integers m. Let A be an $m \times n$ matrix. Let M be the collection of all $n \times m$ matrices.

(a) Prove that the the collection Z of all matrices $B \in M$ so that AB is the zero matrix is a subspace of M.

Solution: We need to show the zero matrix $0_{n\times m}$ is in Z and that Z is closed under addition and scalar multiplication.
To see $0_{n\times m} \in Z$ observe $A0_{n\times m} = 0_{m\times m}$.
To see that Z is closed under addition, let $B_1, B_2 \in Z$. We need to show $B_1 + B_2 \in Z$. Since $B_1, B_2 \in Z$ we know $AB_1 = 0_{m\times m}$ and $AB_2 = 0_{m\times m}$. By the distributive property of matrix multiplication,

$$A(B_1 + B_2) = AB_1 + AB_2 = 0_{m\times m} + 0_{m\times m} = 0_{m\times m}.$$

To see that Z is closed under scalar multiplication, let $B \in Z$ and let $c \in \mathbb{R}$. We must show that $cB \in Z$. Since $B \in Z$ we know $AB = 0_{m\times m}$. Then,

$$A(cB) = c(AB) = c0_{m\times m} = 0_{m\times m}.$$

(b) The collection Z' of all matrices $B \in M$ so that BA is the zero matrix is also a subspace of M. Are these subspaces equal? If they are, then explain why. If not, then give a counterexample consisting of an explicit choice of m and n and a matrix A so that $Z \neq Z'$.

Solution: In general $Z \neq Z'$. For example, suppose $m = n = 2$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
Now let B be a general 2×2 matrix, say $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$AB = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \quad \text{and so} \quad Z = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ c = d = 0\}.$$

We can also compute

$$BA = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \quad \text{and so} \quad Z' = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a = c = 0\}.$$

You can see that $Z \neq Z'$. For example, the matrix $\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$ is in Z but not in Z'.

11. Find a basis for the orthogonal complement to $span\{\mathbf{v}\}$ in $\mathbb{R}^3$ where $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

Solution: Observe $span\{\mathbf{v}\}$ is the row space of the matrix $A = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$, i.e., $span\{\mathbf{v}\} = C(A^T)$. The matrix is already in reduced echelon form. By the Fundamental Theorem of Linear Algebra II, we know the orthogonal complement to the row space is the null space. So, we need to find a basis for $N(A)$.

Observe that A is already in reduced echelon form. There is a pivot in the first column, and the other columns correspond to free variables. The equation represented by the single row of A in the equation $A\mathbf{x} = \mathbf{0}$ is

$$x_1 + 2x_2 + 4x_3 = 0.$$

Solving for the dependent variable x_1 yields

$$x_1 = -2x_2 - 4x_3.$$

Then

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 4x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

A basis for $N(A)$ is given by $\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}\}$.

12. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if $C(A) = \mathbb{R}^n$.

Solution: We have to prove two statements:

1. If A is invertible, then $C(A) = \mathbb{R}^n$.
2. If $C(A) = \mathbb{R}^n$, then A is invertible.

To prove statement (1) is true, assume A is invertible. We will prove $C(A) = \mathbb{R}^n$. To prove this we need to show that for any $\mathbf{b} \in \mathbb{R}^n$, we have $\mathbf{b} \in C(A)$. Recall that $C(A)$ is the collection of linear combinations of the columns, and matrix-vector multiplication gives a linear combination of the columns. So, we need to show $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$. Since A is invertible, it has an inverse A^{-1}. Set $\mathbf{x} = A^{-1}\mathbf{b}$. Then

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b},$$

so we have found a solution.

To prove statement (2) is true, assume $C(A) = \mathbb{R}^n$. This means that $\dim C(A) = \dim \mathbb{R}^n = n$. Since $\dim C(A) = n$, this is the rank of A. That is A has n pivots. Let R be the reduced echelon form of A. Since R is $n \times n$, we know R has a pivot in each row and column. Thus $R = I$. Since $A \sim I$, there is an invertible matrix E so that $EA = I$. Then left multiplying by E^{-1} we see $A = E^{-1}$. This means that A is invertible (and its inverse is E).

Remark: Statement (1) can also be proved using row reduction. Suppose A is invertible. Then A^{-1} exists. Let $B = A^T$ so that $B^{-1} = (A^{-1})^T$. Then the column space of A equals the row space of B. Now recall that the row spaces of all matrices obtained by row reduction are the same. Since B is invertible, we can left multiply by the inverse matrix B^{-1} to see that B row reduces to I. Thus, the row space of B and the row space of I are identical. But, the row space of I is clearly $\mathbb{R}^n$ (and the rows form the standard basis for $\mathbb{R}^n$). Thus the column space of A is $\mathbb{R}^n$ as well.