Math 346: Practice for Midterm 2

Prof. Hooper

Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §3.1-3.5 and §4.1. Knowledge of earlier material will also be necessary to do well on the test, but earlier material will not be explicitly tested. You are expected to know all material covered in the course up until now.

Basic concepts. You should understand and be able to work with basic terms used in the study of Linear Algebra. You should also be able to define most of these terms which are discussed by the book and also were discussed in class.

vector space (p. 123), *subspace* (p. 125), *column space* (p. 127), *span* (p. 128 & 167), *nullspace* (p. 135), *special solution* (p. 135), *reduced row echelon form* (p. 137), *rank* (p. 139), *complete solution* (p. 153), *linearly independent* (p. 165), *row space* (p. 168), *basis* (p. 168), *standard basis* (p. 169), *dimension of a space* (p. 171), *left null space* (p. 181), *Fundamental Theorem of Linear Algebra, Part 1* (p. 185) *orthogonal subspaces* (p. 195) *orthogonal complement* (p. 195) *Fundamental Theorem of Linear Algebra, Part 1* (p. 198)

Techniques. You should be able to:

- Prove that a subset of a vector space is a subspace.
- Reduce a matrix to reduced row echelon form.
- Find complete solutions to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ and make use of the relationships between these equations.
- Find the rank of a matrix. Understand how it relates to the dimensions of the four fundamental subspaces: the column space, null space, row space, and left null space of a matrix.
- Find bases for the four fundamental subspaces. Find bases for subspaces given in other ways (such as by a span or zero sets to linear equations).
- Demonstrate that two subspaces are orthogonal. Compute the orthogonal complement of a subspace.

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test, and they do not cover all possible problems I would ask on a test. You should also make sure you know how to do all homework problems that were assigned!

1. Suppose V is a vector space.
   (a) Complete the following definition:
   A set vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is called *linearly independent* if ...
   (b) Suppose the list of three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent. Prove that the system of two vectors $\mathbf{v}_1, \mathbf{v}_2$ is linearly independent.

2. Let V be a vector space.

   (a) Complete the following definition:
   A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in V spans V if ...

   (b) Suppose that the set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\} \subset V$ is linear independent but does not span V. Fill in the blanks to complete the proof that you can find another vector $\mathbf{v}_3 \in V$ so that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent.

   Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2$ be a system which is linear independent but does not generate V. Since $\mathbf{v}_1, \mathbf{v}_2$ does not generate, there is a vector $\mathbf{v}_3$ such that

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   We claim that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent. Suppose to the contrary that this system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linear dependent. Then, there would be scalars $c_1, c, c_3 \in \mathbb{R}$ which are not all zero and satisfy

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   We will prove that this causes a contradiction. First if $\alpha_3 = 0$, then we have a contradiction because

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   Second if $\alpha_3 \neq 0$, then we have a contradiction because

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   Since one of these two possibilities must occur, it can not be that the system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent. Therefore, we have shown that this system is linearly independent.

3. Let A be an $m \times n$ matrix. Complete the following definitions.

   (a) The *column space* of A is ...

   (b) Recall that the *null space* $N(A)$ is the set of vectors $\mathbf{v} \in \mathbb{R}^n$ so that $A\mathbf{v} = \mathbf{0}$. Prove that $N(A)$ is a subspace of $\mathbb{R}^n$.

4. True or False.

   (a) If A is a square matrix with linearly independent columns, then A is invertible.

   (b) Five vectors in $\mathbb{R}^4$ can span $\mathbb{R}^4$.

   (c) You can find five linearly independent vectors in $\mathbb{R}^4$.

   (d) If the subspaces S and T of $\mathbb{R}^n$ share a common non-zero vector $\mathbf{v}$, then S and T are not orthogonal.

   (e) Two planar subspaces of $\mathbb{R}^3$ can be orthogonal.

5. Find a basis for the column space of

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}.$$

What is the rank of A? How about the dimension of the null space $N(A)$?

6. Find a basis for the plane of solutions to the equation $x + 2y + 3z = 0$.

7. Suppose A is a 4×5 matrix and the reduced row echelon form of A is given by

$$R = \begin{pmatrix} 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for $N(A)$.

(b) Suppose $A\mathbf{x}_p = \mathbf{b}$ where

$$\mathbf{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

8. Suppose the matrix A is 9×7 and has rank 4. What are the dimensions of $C(A)$, $C(A^T)$, $N(A)$ and $N(A^T)$?

9. Let A be an $m \times n$ matrix. Prove that the row space and null spaces of A are orthogonal.

10. Fix positive integers m. Let A be an $m \times n$ matrix. Let M be the collection of all $n \times m$ matrices.

(a) Prove that the the collection Z of all matrices $B \in M$ so that AB is the zero matrix is a subspace of M.

(b) The collection Z' of all matrices $B \in M$ so that BA is the zero matrix is also a subspace of M. Are these subspaces equal? If they are, then explain why. If not, then give a counterexample consisting of an explicit choice of m and n and a matrix A so that $Z \neq Z'$.

11. Find a basis for the orthogonal complement to $span\{\mathbf{v}\}$ in $\mathbb{R}^3$ where $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

12. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if $C(A) = \mathbb{R}^n$.