Math 346: Practice for Midterm 1

Prof. Hooper

Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §1.1-1.3, §2.1-2.6, §8.1 and the two handouts available on the course website.

Basic concepts. You should understand and be able to work with basic terms used in the study of Linear Algebra. You should also be able to define most of these terms which are discussed by the book and also were discussed in class.

linear combination (p. 1-3), dot product of vectors (p. 11), length of a vector (p. 12), unit vector (p. 13, 14), matrix-vector multiplication (p. 22, 36-37), coefficient matrix (p. 33, 36), back substitution (p. 34), elimination or row reduction (p. 46), elementary or elimination matrix (p. 60), matrix multiplication (p. 61), augmented matrix (p. 63), inverse matrix (p.83, handout 2), upper triangular matrix (p. 46, 97), lower triangular matrix (p. 98), triangular matrix (p. 52, 89), LU-factorization (p. 97), Linear transformation (p. 401, handout 1), composition (handout 1), identity transformation (p. 402), matrix of a linear transformation (handout 1), identity matrix (p. 37, handout 1)

Techniques. You should be able to:

- Apply elimination to a matrix, reducing it to upper triangular form or the identity matrix. Recognize pivots in the reduced matrix and understand their meaning.
- Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ in its various forms (system of linear equations, matrix equation, solution to linear combination problem).
- Demonstrate an understanding of linear transformations. You should be able to use the definition of *linear transformation* and manipulate linear transformations. Find the matrix of a linear transformation.
- Decide if a matrix is invertible and find its inverse if it is invertible. Write an invertible matrix as a product of elementary (or elimination) matrices .
- Factor a matrix A into the form LU, where L is a lower triangular matrix and U is an upper triangular matrix. You should be able to use this factorization to solve the equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test, and they do not cover all possible problems I would ask on a test. You should also make sure you know how to do all homework problems that were assigned!

1. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

For each of the following expressions, either evaluate the expression or state that it is undefined.

(a) $A + B$

Solution: Undefined.

(b) $2B$

Solution: $2B = \begin{pmatrix} 4 & -2 \\ 10 & 6 \end{pmatrix}$.

(c) AB

Solution: $AB = \begin{pmatrix} 12 & 5 \\ 13 & 10 \\ 13 & -1 \end{pmatrix}$.

(d) BA

Solution: Undefined

(e) $A\mathbf{v}$

Solution: $A\mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix}$.

2. Observe that the vectors $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ have the property that the sum of the entries of the vectors is zero.

(a) Complete the following definition:
The vector $\mathbf{x}$ *is a linear combination of* $\mathbf{v}$ *and* $\mathbf{w}$ *if . . .*

Solution: there are real numbers c and d so that $\mathbf{x} = c\mathbf{v} + d\mathbf{w}$.

(b) Show that if $\mathbf{x}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$, then the sum of the entries of $\mathbf{x}$ is zero.

Solution: Suppose $\mathbf{x} = c\mathbf{v} + d\mathbf{w}$ for some real numbers c and d. Then

$$\mathbf{x} = c\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ d \\ -d \end{pmatrix} = \begin{pmatrix} c \\ d-c \\ -d \end{pmatrix}.$$

So the sum of the entries is

$$c + (d - c) + (-d) = 0.$$

(c) Show that any $\mathbf{y} \in \mathbb{R}^3$ whose entries sum to zero can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. (*Hint*: If the first two entries of $\mathbf{y}$ are y_1 and y_2, then the third must be $-y_1 - y_2$.)

Solution: Suppose the entries of $\mathbf{y}$ sum to zero. Then we have

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ -y_1 - y_2 \end{pmatrix}$$

as suggested in the hint. We want to find c and d so that $\mathbf{y} = c\mathbf{v} + d\mathbf{w}$. Above we showed

$$c\mathbf{v} + d\mathbf{w} = \begin{pmatrix} c \\ d - c \\ -d \end{pmatrix}.$$

Thus we must have $c = y_1$ and $d = y_1 + y_2$. We will check that this works:

$$y_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + (y_1 + y_2) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} y_1 \\ -y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1 + y2 \\ -y_1 - y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ -y_1 - y_2 \end{pmatrix} = \mathbf{y}.$$

3. Let $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (1, 0, 1)$.

(a) What are the lengths of $\mathbf{v}$ and $\mathbf{w}$?

Solution: The length of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}.$$

Similarly $\|\mathbf{w}\| = \sqrt{2}$.

(b) What is their dot product?

Solution: We have

$$\mathbf{v} \cdot \mathbf{w} = 1 * 1 + 1 * 0 + 0 * 1 = 1.$$

(c) What is the angle between the vectors?

Solution: By the cosine formula, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ satisfies

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

This means that $\theta = \frac{\pi}{3}$ radians (or 60 degrees) since $\cos\frac{\pi}{3} = \frac{1}{2}$.

(d) How does computing the length of $\mathbf{v} - \mathbf{w}$ demonstrate that your answer to part (c) is correct?

Solution: We have $\mathbf{v} - \mathbf{w} = (0, 1, -1)$ which is also length $\sqrt{2}$. The vectors $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ represent edges between the three points consisting of the origin $\mathbf{0} = (0, 0, 0)$, $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (1, 0, 1)$. Because the lengths of all three edges are equal, the triangle is equilateral. All angles of an equilateral triangle are $\frac{\pi}{3}$ (or 60 degrees).

4. Solve each of the linear systems below using elimination. Explicitly describe all solutions.

(a) $\begin{cases} 3x - 5y &= 1 \\ -6x + 10y &= 1. \end{cases}$

Solution: The augmented matrix is

$$\left(\begin{array}{rr|r} 3 & -5 & 1 \\ -6 & 10 & 1 \end{array}\right).$$

Now we row reduce. We add 2 copies of the first row to the second.

$$\left(\begin{array}{rr|r} 3 & -5 & 1 \\ -6 & 10 & 1 \end{array}\right) \sim \left(\begin{array}{rrr} 3 & -5 & 1 \\ 0 & 0 & 3 \end{array}\right).$$

At this point, we can recognize that there is $\boxed{\textit{no solution}}$. (The last line of the reduced matrix corresponds to the equation $0 = 3$.)

(b) $\begin{cases} x + 5z &= 4 \\ 2x - y + 7z &= 11 \\ x - 3y - 4z &= 13. \end{cases}$

Solution: The augmented matrix is given by

$$\left(\begin{array}{rrr|r} 1 & 0 & 5 & 4 \\ 2 & -1 & 7 & 11 \\ 1 & -3 & -4 & 13 \end{array}\right).$$

We row reduce. First we subtract two copies of the first row from the second row and subtract one copy of the first row from the third row. Finally we subtract three copies of the second row to the third row.

$$\left(\begin{array}{rrr|r} 1 & 0 & 5 & 4 \\ 2 & -1 & 7 & 11 \\ 1 & -3 & -4 & 13 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 5 & 4 \\ 0 & -1 & -3 & 3 \\ 0 & -3 & -9 & 9 \end{array}\right) \sim \left(\begin{array}{rrrr} 1 & 0 & 5 & 4 \\ 0 & -1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We observe that x and y are basic variables and z is a free variable. (There is no pivot in the column associated to z.) We can find all solutions using back substitution. The system corresponding to the reduced matrix is

$$\begin{cases} x + 5z &= 4 \\ -y - 3z &= 3 \\ 0 &= 0. \end{cases}$$

Solving for y we see $y = -3z - 3$. Then solving for x we see $x = 4 - 5z$. The set of all solutions to the equation is then given by the set of vectors of the form $\mathbf{v} = (x, y, z)$:

$$\left\{ \begin{pmatrix} 4 - 5z \\ -3 - 3z \\ z \end{pmatrix} \;:\; z \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + z \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} \;:\; z \in \mathbb{R} \right\}.$$

5. True or false:

(a) If $T_1 : \mathbb{R}^n \to \mathbb{R}^p$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^n$ are linear transformations, then the composition $T_1 \circ T_2$ is a linear transformation.

Solution: True. We proved this in class.

(b) A linear system with n equations and n unknowns always has a solution.

Solution: False. For example, the linear system

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 3 \end{cases}$$

has two equations and two unknowns and two unknowns but no solution.

(c) If A and B are invertible matrices, then the product AB is also invertible.

Solution: True: $(AB)^{-1} = B^{-1}A^{-1}$.

(d) Any matrix with a left inverse also has a right inverse.

Solution: False. For example, the matrix $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$ has a right inverse, for example

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

but has no left inverse. The fact that it has no left inverse can be seen by multiplying with a general matrix 1×2 matrix:

$$\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} a & a \\ c & c \end{pmatrix}.$$

Observe that there is no way the matrix at right could be the identity.

6. (a) Let V and W be vector spaces. A transformation $T : V \to W$ is called *linear* if ...

Solution: The following two statements hold:

1. For all vectors $\mathbf{v}$ and $\mathbf{w}$, $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$.

2. For all vectors $\mathbf{v}$ and scalars s, $F(s\mathbf{v}) = sF(\mathbf{v})$.

Another possible answer is:

$$F(a\mathbf{v} + b\mathbf{w}) = aF(\mathbf{v}) + bF(\mathbf{w})$$

for all vectors $\mathbf{v}$ and $\mathbf{w}$ and all scalars a and b.

(b) Consider the transformation $F : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y.$$

Prove that this map is not linear. (*Hint:* Let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If it were linear then how would $F(2\mathbf{v})$ relate to $F(\mathbf{v})$?)

Solution: We'll follow the hint setting $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Observe

$$F(\mathbf{v}) = F\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1^2 + 1 = 2.$$

Also

$$F(2\mathbf{v}) = F\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2^2 + 2 = 6.$$

If it were linear then $F(2\mathbf{v})$ would equal $2F(\mathbf{v})$ which in this case is $2 * 2 = 4$. So, the map can not be linear.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ 3x \end{pmatrix}.$$

Find the matrix of T. Check that applying T is the same as multiplication by this matrix.

Solution: The columns of the matrix are the images of the columns of the identity matrix. So, the first column of the matrix is

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

The second column will be

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Therefore the matrix of T is

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}.$$

We will now perform the check:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ 3x \end{pmatrix}.$$

7. Let $A = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

(a) Find A^{-1}.

Solution: To find A^{-1}, we will row reduce $(A|I)$ to $(A^{-1}|I)$. First we write down $(A|I)$, then we swap the first two rows, and add -2 times the first row to the third row.

$$(A|I) = \left(\begin{array}{ccc|ccc} 0 & 3 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 \end{array}\right).$$

Now we add one copy of the third row to the first and add three copies to the second row. Then we negate the last row.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 & -6 & 3 \\ 0 & -1 & 0 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 & -6 & 3 \\ 0 & 1 & 0 & 0 & 2 & -1 \end{array}\right).$$

Now we divide the second row by 2 and swap it with the third row. After this, we subtract the third row from the first.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 & -6 & 3 \\ 0 & 1 & 0 & 0 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -3 & \frac{3}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{2} & 2 & \frac{-1}{2} \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -3 & \frac{3}{2} \end{array}\right).$$

Therefore, the answer is

$$A^{-1} = \begin{pmatrix} \frac{-1}{2} & 2 & \frac{-1}{2} \\ 0 & 2 & -1 \\ \frac{1}{2} & -3 & \frac{3}{2} \end{pmatrix}.$$

(b) There is a matrix B so that $AB = \begin{pmatrix} 4 & 2 \\ 1 & 1 \\ 2 & 4 \end{pmatrix}$. Find B.

Solution: By left multiplication by A^{-1}, we see that

$$B = A^{-1}\begin{pmatrix} 4 & 2 \\ 1 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} & 2 & \frac{-1}{2} \\ 0 & 2 & -1 \\ \frac{1}{2} & -3 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -2 \\ 2 & 4 \end{pmatrix}.$$

8. Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 7 & 6 \\ 6 & 8 & 9 \end{pmatrix}$.

(a) Find a lower triangular matrix L and an upper triangular matrix U so that $A = LU$.

Solution: We perform the row reduction:

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 7 & 6 \\ 6 & 8 & 9 \end{pmatrix} \sim_1 \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & -1 & -3 \end{pmatrix} \sim_2 \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{pmatrix} = U.$$

The steps in the row reduction were:

1. Subtract $2 * row1$ from $row2$ and subtract $3 * row1$ from $row3$.
2. Add $row2$ to $row3$.

The negations of these steps are:

1. Add $2 * row1$ to $row2$ and add $3 * row1$ to $row3$.
2. Subtract $row2$ from $row3$.

We apply these in reverse order to I to find L.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim_2^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \sim_1^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} = L.$$

We will now check our work:

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 7 & 6 \\ 6 & 8 & 9 \end{pmatrix} = A.$$

(b) Use your factorization and forward and back substitution to solve $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Above we have written $A = LU$, so we want to find an $\mathbf{x}$ so that $LU\mathbf{x} = \mathbf{b}$. We do this by first finding a $\mathbf{c}$ so that $L\mathbf{c} = \mathbf{b}$ and then finding $\mathbf{x}$ so that $U\mathbf{x} = \mathbf{c}$. Then $A\mathbf{x} = LU\mathbf{x} = L\mathbf{c} = \mathbf{b}$.

So, first we need to solve $L\mathbf{c} = \mathbf{b}$. Let $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. Then

$$L\mathbf{c} = \begin{pmatrix} c_1 \\ 2c_1 + c_2 \\ 3c_1 - c_2 + c_3 \end{pmatrix}.$$

We need this to equal $\mathbf{b}$ so

$$c_1 = 1, \quad 2c_1 + c_2 = 1, \quad \text{and} \quad 3c_1 - c_2 + c_3 = 1.$$

By forward substitution we see $c_1 = 1$, $c_2 = -1$ and $c_3 = -3$. So $\mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$.

Now we solve $U\mathbf{x} = \mathbf{c}$ for $\mathbf{x}$ using back substitution. Writing $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, we have

$$U\mathbf{x} = \begin{pmatrix} 2x_1 + 3x_2 + 4x_3 \\ x_2 - 2x_3 \\ -5x_3 \end{pmatrix}.$$

Thus

$$2x_1 + 3x_2 + 4x_3 = 1, \quad x_2 - 2x_3 = -1, \quad \text{and} - 5x_3 = -3.$$

By back substitution, we see $x_3 = \frac{3}{5}$, $x_2 = \frac{1}{5}$ and $x_1 = -1$.

9. (a) Complete the following definition:
An $n \times n$ matrix A is invertible if ...

Solution: there is an matrix A^{-1} so that $AA^{-1} = I$ and $A^{-1}A = I$.

(b) Suppose there is a a vector $\mathbf{x}$ so that $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$. Explain why A is not invertible.

Solution: We will explain why it is impossible for there to be an $\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$ so that $A\mathbf{x} = \mathbf{0}$ and for A to also be invertible. Assume all these statements are true. Then A has an inverse matrix A^{-1}. Since we know $A\mathbf{x} = \mathbf{0}$ we know by left multiplication by A^{-1} on each side that $A^{-1}A\mathbf{x} = A^{-1}\mathbf{0}$. But, $A^{-1}A = I$ and $I\mathbf{x} = \mathbf{x}$. On the other hand, any matrix times the zero vector is zero. Thus,

$$\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}.$$

We have shown that $\mathbf{x} = \mathbf{0}$ but we also know $\mathbf{x} \neq \mathbf{0}$. This is a contradiction, so A could not have been invertible.

10. Recall that an elimination step represents left multiplication by an invertible (*elementary*) matrix. Suppose A is a matrix with 3 rows.

(a) Find a matrix E so that EA is obtained from A by adding twice row 1 to row 3.

Solution:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

(b) Find a matrix P so that PA is obtained from A by swapping row 1 and row 2.

Solution:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Find a matrix D so that DA is obtained from A by tripling row 3.

Solution:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(d) Applying the above elimination steps (in order *abc*) reduces the matrix B to the identity. Write B as a product of elementary (or *elimination*) matrices.

Solution: Since applying the elimination steps to B reduces B to I, we know that $DPEB = I$. Thus $B = E^{-1}P^{-1}D^{-1}$. Writing the inverses of these matrices yields:

$$B = E^{-1}P^{-1}D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$