Math 346: Practice for Midterm 1

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §1.1-1.3, §2.1-2.6, §8.1 and the two handouts available on the course website.

Basic concepts. You should understand and be able to work with basic terms used in the study of Linear Algebra. You should also be able to define most of these terms which are discussed by the book and also were discussed in class.

linear combination (p. 1-3), dot product of vectors (p. 11), length of a vector (p. 12), unit vector (p. 13, 14), matrix-vector multiplication (p. 22, 36-37), coefficient matrix (p. 33, 36), back substitution (p. 34), elimination or row reduction (p. 46), elementary or elimination matrix (p. 60), matrix multiplication (p. 61), augmented matrix (p. 63), inverse matrix (p.83, handout 2), upper triangular matrix (p. 46, 97), lower triangular matrix (p. 98), triangular matrix (p. 52, 89), LU-factorization (p. 97), Linear transformation (p. 401, handout 1), composition (handout 1), identity transformation (p. 402), matrix of a linear transformation (handout 1), identity matrix (p. 37, handout 1)

Techniques. You should be able to:

- Apply elimination to a matrix, reducing it to upper triangular form or the identity matrix. Recognize pivots in the reduced matrix and understand their meaning.
- Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ in its various forms (system of linear equations, matrix equation, solution to linear combination problem).
- Demonstrate an understanding of linear transformations. You should be able to use the definition of *linear transformation* and manipulate linear transformations. Find the matrix of a linear transformation.
- Decide if a matrix is invertible and find its inverse if it is invertible. Write an invertible matrix as a product of elementary (or elimination) matrices .
- Factor a matrix A into the form LU, where L is a lower triangular matrix and U is an upper triangular matrix. You should be able to use this factorization to solve the equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test, and they do not cover all possible problems I would ask on a test. You should also make sure you know how to do all homework problems that were assigned!

1. Let
$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   For each of the following expressions, either evaluate the expression or state that it is undefined.

(a) $A + B$

(b) $2B$

(c) AB

(d) BA

(e) $A\mathbf{v}$

2. Observe that the vectors $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ have the property that the sum of the entries of the vectors is zero.

   (a) Complete the following definition:
   The vector $\mathbf{x}$ *is a linear combination of* $\mathbf{v}$ *and* $\mathbf{w}$ *if ...*

   (b) Show that if $\mathbf{x}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$, then the sum of the entries of $\mathbf{x}$ is zero.

   (c) Show that any $\mathbf{y} \in \mathbb{R}^3$ whose entries sum to zero can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. (*Hint*: If the first two entries of $\mathbf{y}$ are y_1 and y_2, then the third must be $-y_1 - y_2$.)

3. Let $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (1, 0, 1)$.

   (a) What are the lengths of $\mathbf{v}$ and $\mathbf{w}$?

   (b) What is their dot product?

   (c) What is the angle between the vectors?

   (d) How does computing the length of $\mathbf{v} - \mathbf{w}$ demonstrate that your answer to part (c) is correct?

4. Solve each of the linear systems below using elimination. Explicitly describe all solutions.

   (a) $\begin{cases} 3x - 5y &= 1 \\ -6x + 10y &= 1. \end{cases}$

   (b) $\begin{cases} x + 5z &= 4 \\ 2x - y + 7z &= 11 \\ x - 3y - 4z &= 13. \end{cases}$

5. True or false:

   (a) If $T_1 : \mathbb{R}^n \to \mathbb{R}^p$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^n$ are linear transformations, then the composition $T_1 \circ T_2$ is a linear transformation.

   (b) A linear system with n equations and n unknowns always has a solution.

   (c) If A and B are invertible matrices, then the product AB is also invertible.

   (d) Any matrix with a left inverse also has a right inverse.

6. (a) Let V and W be vector spaces. A transformation $T : V \to W$ is called *linear* if ...

   (b) Consider the transformation $F : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$F \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y.$$

   Prove that this map is not linear. (*Hint:* Let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If it were linear then how would $F(2\mathbf{v})$ relate to $F(\mathbf{v})$?)

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ 3x \end{pmatrix}.$$

Find the matrix of T. Check that applying T is the same as multiplication by this matrix.

7. Let $A = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

   (a) Find A^{-1}.

   (b) There is a matrix B so that $AB = \begin{pmatrix} 4 & 2 \\ 1 & 1 \\ 2 & 4 \end{pmatrix}$. Find B.

8. Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 7 & 6 \\ 6 & 8 & 9 \end{pmatrix}$.

   (a) Find a lower triangular matrix L and an upper triangular matrix U so that $A = LU$.

   (b) Use your factorization and forward and back substitution to solve $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

9. (a) Complete the following definition:
   An $n \times n$ matrix A is invertible if ...

   (b) Suppose there is a a vector $\mathbf{x}$ so that $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$. Explain why A is not invertible.

10. Recall that an elimination step represents left multiplication by an invertible (*elementary*) matrix. Suppose A is a matrix with 3 rows.

    (a) Find a matrix E so that EA is obtained from A by adding twice row 1 to row 3.

    (b) Find a matrix P so that PA is obtained from A by swapping row 1 and row 2.

    (c) Find a matrix D so that DA is obtained from A by tripling row 3.

    (d) Applying the above elimination steps (in order *abc*) reduces the matrix B to the identity. Write B as a product of elementary (or *elimination*) matrices.