Math 346: Linear Algebra: Second Midterm Solutions

Monday Nov. 6th, 2017

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1. Suppose that A is an $m \times n$ matrix with rank r.

   (a) (3 points) What is the dimension of the column space of A?

   > **Solution:** $\dim C(A) = r$.

   (b) (3 points) What is the dimension of the row space of A?

   > **Solution:** $\dim C(A^T) = r$.

   (c) (3 points) What is the dimension of the null space of A?

   > **Solution:** $\dim N(A) = n - r$.

   (d) (3 points) What is the dimension of the left null space of A?

   > **Solution:** $\dim N(A^T) = m - r$.

   (e) (3 points) What is the dimension of the orthogonal compliment of the row space of A?

   > **Solution:** The orthogonal complement of the row space of A is the null space, which has dimension $n - r$.

2. The matrix A is given below along with its reduced echelon form R:

$$A = \begin{pmatrix} 2 & -6 & 4 & 1 & 3 \\ -1 & 3 & -2 & 2 & 3 \\ 0 & 0 & 0 & 3 & -1 \\ -3 & 9 & -6 & 1 & -3 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   (a) (3 points) What is the rank of A?

Solution: The rank of A is 3 because there are three pivots.

(b) (4 points) Find a basis for the column space $C(A)$.

Solution: The pivot columns in the original matrix A form a basis for $C(A)$. The pivot columns are columns 1, 4 and 5 so $\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -1 \\ -3 \end{pmatrix} \right\}$.

(c) (4 points) Find a basis for the row space $C(A^T)$.

Solution: The non-zero rows of the row reduced form of A form a basis for $C(A^T)$, so in this case we get the basis

$$\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(d) (6 points) Find a basis for the null space $N(A)$.

Solution: Recall that $N(A) = N(R)$ since A reduces to R using elimination. The null space $N(R)$ is the collection of solutions to $R\mathbf{x} = \mathbf{0}$. The rows of R correspond to the equations

$$x_1 - 3x_2 + 2x_3 = 0, \quad x_4 = 0, \quad x_5 = 0, \quad \text{and} \quad 0 = 0.$$

Solving for the dependent variables x_1, x_4 and x_5 in terms of the free variables x_2 and x_3 yields

$$x_1 = 3x_2 - 2x_3, \quad x_4 = 0, \quad \text{and} \quad x_5 = 0.$$

Then we can write the solutions to $R\mathbf{x} = \mathbf{0}$ (or $A\mathbf{x} = \mathbf{0}$) as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus a basis for $N(A)$ is given by $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

3. (10 points) Let $n \geq 1$ and let A be an $n \times n$ matrix. Prove that the set of vectors $\mathbf{v} \in \mathbb{R}^n$ so that $A\mathbf{v} = 3\mathbf{v}$ is a subspace of $\mathbb{R}^n$.

Solution: Let S be the set of vectors $\mathbf{v} \in \mathbb{R}^n$ satisfying $A\mathbf{v} = 3\mathbf{v}$. We need to check that $\mathbf{0} \in S$ and that S is closed under addition and scalar multiplication.

Observe that $\mathbf{0} \in S$ because $A\mathbf{0} = \mathbf{0}$ and $3\mathbf{0} = \mathbf{0}$.

To see that S is closed under addition assume that $\mathbf{v}_1, \mathbf{v}_2 \in S$. We must show that $\mathbf{v}_1 + \mathbf{v}_2 \in S$. Since $\mathbf{v}_1, \mathbf{v}_2 \in S$ we know that $A\mathbf{v}_1 = 3\mathbf{v}_1$ and $A\mathbf{v}_2 = 3\mathbf{v}_2$. Then,

$$A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2 = 3\mathbf{v}_1 + 3\mathbf{v}_2 = 3(\mathbf{v}_1 + \mathbf{v}_2)$$

which demonstrates that $\mathbf{v}_1 + \mathbf{v}_2 \in S$.

To see that S is closed under scalar multiplication, let $\mathbf{v} \in S$ and $c \in \mathbb{R}$. We must show that $c\mathbf{v} \in S$. Since $\mathbf{v} \in S$ we know that $A\mathbf{v} = 3\mathbf{v}$. Then,

$$A(c\mathbf{v}) = cA\mathbf{v} = c(3\mathbf{v}) = 3(c\mathbf{v}).$$

This shows that $c\mathbf{v} \in S$.

4. State if the statement is True or False. Give a brief justification for your answer.

(a) (4 points) The collection of three vectors $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ spans $\mathbb{R}^2$.

Solution: True, because $\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (They do not form a basis.)

(b) (4 points) The equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if $\mathbf{b} \in C(A)$.

Solution: True. Actually "if" above can be replaced with "if and only if" because $C(A)$ consists of vectors which are a linear combination of the columns, and $A\mathbf{x}$ describes a linear combination of the columns of A.

(c) (4 points) A vector space can consist of exactly two vectors.

Solution: False. One of the vectors must not be the zero vector. Call this vector $\mathbf{v}$. Then the vector space has to contain the vectors $c\mathbf{v}$ for each $c \in \mathbb{R}$.

(d) (4 points) The column space of a matrix is always orthogonal to the null space.

Solution: False. If the matrix is $m \times n$ then $N(A) \subset \mathbb{R}^n$ and $C(A) \subset \mathbb{R}^m$. So, in general they are not even subspaces of the same vector space!

(e) (4 points) If AB is the zero matrix then the row space of A is orthogonal to the column space of B.

Solution: True. An element of the row space of A has the form $A^T\mathbf{y}$. An element of the column space of B has the form $B\mathbf{x}$. Then assuming $AB = Z$ where Z is the zero matrix, we have:

$$(A^T\mathbf{y}) \cdot (B\mathbf{x}) = (A^T\mathbf{y})^T B\mathbf{x} = \mathbf{y}^T AB\mathbf{x} = \mathbf{y}^T Z\mathbf{x} = \mathbf{y}^T\mathbf{0} = 0.$$

Alternate view: We can also argue that the entries of AB are the dot products of the rows of A with the columns of B. Since the row of A span the row space of A, and the columns of B span the column space of B, it follows that the spaces are orthogonal. (It is enough to check orthogonality on pairs of vectors coming from spanning sets.)

(f) (4 points) If A is an $m \times n$ matrix and B is a $n \times p$ matrix, then any vector in $N(B)$ is also in $N(AB)$.

Solution: True. If $\mathbf{x} \in N(B)$ then $B\mathbf{x} = \mathbf{0}$. Then, $AB\mathbf{x} = \mathbf{0}$ also. Thus $\mathbf{x} \in N(AB)$.

5. (a) (5 points) Complete the following definition:
The span of a collection $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *of vectors is ...*

Solution: The set of vectors of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

for some choice of scalars $c_1, \ldots, c_k \in \mathbb{R}$.

(b) (5 points) Complete the following definition:
Two subspaces V *and* W *of* $\mathbb{R}^n$ *are* orthogonal *if ...*

> **Solution:** $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{v} \in V$ and every $w \in W$.

(c) (10 points) Let $V = span\ \{\mathbf{v}_1\}$ and $W = span\ \{\mathbf{w}_1, \mathbf{w}_2\}$ where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Show that the subspaces V and W of $\mathbb{R}^3$ are orthogonal.

> **Solution:** A general element of V has the form
>
> $$\mathbf{v} = c\mathbf{v}_1 = \begin{pmatrix} c \\ c \\ c \end{pmatrix}$$
>
> for some $c \in \mathbb{R}$. A general element of W has the form
>
> $$\mathbf{w} = d\mathbf{w}_1 + e\mathbf{w}_2 = \begin{pmatrix} d \\ e \\ -d-e \end{pmatrix}$$
>
> for some $d, e \in \mathbb{R}$. Then,
>
> $$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} c \\ c \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ -d-e \end{pmatrix} = cd + ce + c(-d-e) = 0.$$
>
> Since $\mathbf{v} \cdot \mathbf{w}$ for arbitrary $\mathbf{v} \in V$ and $\mathbf{w} \in W$, we know V is orthogonal to W by definition.

6. (a) (5 points) Complete the following definition: The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is *linearly independent* if ...

> **Solution:** the only scalars $c_1, \ldots, c_k \in \mathbb{R}$ so that
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$$
>
> are given by $c_1 = c_2 = \ldots = c_k = 0$.

(b) (10 points) Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 4 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 7 \end{pmatrix}$. Is the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Give an argument or calculation explaining why or why not.

Solution: Let A be the matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Then

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 0 \\ 2 & 4 & 7 \end{pmatrix}.$$

Then linear combinations of the vectors have the form $A\mathbf{x}$, and to show they are linearly independent we must show that $\mathbf{0}$ is the only solution to the equation $A\mathbf{x} = \mathbf{0}$. That is, we must show there are no free variables (or equivalently there is a pivot in every column). We row reduce A below.

First, clear out the entries below the top left entry by adding multiples of the first row to the other rows.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 0 \\ 2 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & 5 \end{pmatrix}.$$

Now we clear out the entries below the entry in row 2 and column 2 by adding multiples of row 2 to the rows below:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 2 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, we can swap row 3 with row 4, and zero out the fourth row by subtracting 3 times row 3:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The final matrix at right above is in echelon form and clearly has a pivot in each column. Thus the columns are linearly independent. (Indeed, the three columns form a basis for the column space.)