Math 346: Linear Algebra: First Midterm Solutions

Wed. Oct. 4th, 2017

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1. (15 points) Find all solutions to the linear system of equations

$$\begin{aligned} 3x - 3y + 15z &= 0 \\ 2x - 3y + 8z &= 0 \\ x - 2y + 3z &= 0. \end{aligned}$$

Solution: We apply elimination to the augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -3 & 15 & 0 \\ 2 & -3 & 8 & 0 \\ 1 & -2 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -3 & 8 & 0 \\ 3 & -3 & 15 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From this we can see the system has the same solution set as the system:

$$\begin{aligned} x - 2y + 3z &= 0 \\ y + 2z &= 0 \\ 0 &= 0. \end{aligned}$$

There is no pivot in column 3, so the variable z is free. The other variables can be expressed as a function of z. Since $y + 2z = 0$ we know $y = -2z$. Also,

$$x = 2y - 3z = -4z - 3z = -7z.$$

It follows that the solution set is given by

$$\{(x, y, z) \ : \ x = -7z \text{ and } y = -2z.\}$$

2. Let x be a real number and $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & x \end{pmatrix}$. (*Remark:* Your answers below should depend on x.)

   (a) (10 points) Find A^{-1}.

Solution: We will apply elimination reducing $(A|I)$ to $(I|A^{-1})$. For the first step, we subtract row 1 from row 2 and row 3:

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & x & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & x-1 & -1 & 0 & 1 \end{array}\right).$$

Next we swap row 2 with row 3:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & x-1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & x-1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right).$$

Now we add row 3 to row 1 and add $x-1$ times row 3 to row 2:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & x-1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -x & x-1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right).$$

Now we add row 2 to row 1:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -x & x-1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -x & x & 1 \\ 0 & -1 & 0 & -x & x-1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right).$$

Finally we negate rows 2 and 3 (scaling both by -1):

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -x & x & 1 \\ 0 & -1 & 0 & -x & x-1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -x & x & 1 \\ 0 & 1 & 0 & x & 1-x & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) = (I|A^{-1}).$$

We have shown that

$$A^{-1} = \begin{pmatrix} -x & x & 1 \\ x & 1-x & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

(b) (6 points) Find the vector $\mathbf{v}$ so that $A\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Solution:

$$\mathbf{v} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -x & x & 1 \\ x & 1-x & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x+3 \\ -x-1 \\ -1 \end{pmatrix}.$$

(c) (6 points) Find the matrix B so that $BA = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Solution: Observe

$$B = (BA)A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -x & x & 1 \\ x & 1-x & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x+1 & -x & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

3. (15 points) Let $A = \begin{pmatrix} 3 & 4 & -5 \\ 3 & 6 & 1 \\ 6 & 4 & -26 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U so that $A = LU$.

Solution: We row reduce to find U:

$$A = \begin{pmatrix} 3 & 4 & -5 \\ 3 & 6 & 1 \\ 6 & 4 & -26 \end{pmatrix} \sim_1 \begin{pmatrix} 3 & 4 & -5 \\ 0 & 2 & 6 \\ 6 & 4 & -26 \end{pmatrix} \sim_2 \begin{pmatrix} 3 & 4 & -5 \\ 0 & 2 & 6 \\ 0 & -4 & -16 \end{pmatrix} \sim_3 \begin{pmatrix} 3 & 4 & -5 \\ 0 & 2 & 6 \\ 0 & 0 & -4 \end{pmatrix} = U.$$

The steps performed above were:

$\sim_1$: Subtract row 1 from row 2.

$\sim_2$: Subtract twice row 1 from row 3.

$\sim_3$: Add twice row 2 to row 3.

To find L we perform the inverse operations in reverse order to I. The inverse operations are:

$\sim_1^{-1}$: Add row 1 to row 2.

$\sim_2^{-1}$: Add twice row 1 to row 3.

$\sim_3^{-1}$: Subtract twice row 2 from row 3.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \sim_2^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \sim_1^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} = L.$$

We check our work:

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & -5 \\ 0 & 2 & 6 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 4 & -5 \\ 3 & 6 & 1 \\ 6 & 4 & -26 \end{pmatrix} = A.$$

4. Suppose A is a 3×3 matrix and $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

   (a) (6 points) Suppose also that $A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. What is $A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$?

   (*Hint*: The map sending $\mathbf{x}$ to $A\mathbf{x}$ is linear.)

   Solution: By linearity, we know that

$$A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}.$$

   (b) (6 points) Suppose in addition $A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$. What is A?

   Solution: The columns of A are $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We already know the first two columns to find the third observe that

$$3A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

   So $A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{-1}{3} \\ \frac{-1}{3} \end{pmatrix}$. Finally as stated in the beginning of the solution, we have

$$A = \left[\, A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad A\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \,\right] = \begin{bmatrix} 2 & 2 & \frac{1}{3} \\ 1 & 4 & \frac{-1}{3} \\ 2 & 4 & \frac{-1}{3} \end{bmatrix}.$$

5. The following shows two elimination steps:

$$\begin{pmatrix} 3 & 6 & 2 \\ -6 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 6 & 2 \\ 0 & 1 & 0 \\ -6 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & 6 & 2 \\ 0 & 1 & 0 \\ 0 & 8 & 5 \end{pmatrix}.$$

   Denote the left matrix above by A, the middle matrix B and the right matrix C.

   (a) (6 points) Which 3×3 matrix P satisfies $PA = B$?

Solution: The matrix P is obtained by swapping row 2 and row 3 of the identity matrix.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) (6 points) Which 3×3 matrix E satisfies $EB = C$? The matrix E is obtained from the identity matrix by adding twice row 1 to row 3.

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

(c) (6 points) Which 3×3 matrix M satisfies $MC = A$?

Solution: We have $PA = B$ so $A = P^{-1}B$. Also since $EB = C$ we know $B = E^{-1}C$. Putting these together we see

$$A = P^{-1}B = P^{-1}E^{-1}C.$$

Therefore we have $M = P^{-1}E^{-1}$. We have

$$P^{-1} = P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

Thus

$$M = P^{-1}E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

6. (8 points) Suppose A and B are $n \times n$ matrices. Expand $(A+2B)(A-2B)$ and simplify so that the result has no parenthesis.

Solution: We have

$$(A + 2B)(A - 2B) = A(A - 2B) + 2B(A - 2B) = A^2 - 2AB + 2BA - 4B^2.$$

7. (10 points) Suppose A and B are $n \times n$ matrices and that B is invertible. Suppose also that $BAB^{-1} = I$. Give a carefully justified argument showing that $A = I$.

Solution: We know that $BAB^{-1} = I$. By right multiplying both sides of this equation by B we see:

$$BAB^{-1}B = IB.$$

We can simplify the left side as $BAB^{-1}B = BAI = BA$, since B and B^{-1} and I is the identity matrix. The right side simplifies to $IB = B$. Thus we see

$$BA = B.$$

Now we multiply the left side both sides of this equation by B^{-1}. We see

$$B^{-1}BA = B^{-1}B.$$

Again, the left side simplifies $B^{-1}BA = IA = A$ and the right side simplifies $B^{-1}B = I$. So we see

$$A = I.$$

Remark: It is tempting but incorrect to simply write $BAB^{-1} = A$. For example, if $BAB^{-1} = C$, it is not generally true that $A = C$.