Complex Diagonalization Example

Problem Diagonalize the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

First we will find the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & -1 \\ 1 & -\lambda & 0 & -1 \\ 0 & 1 & -\lambda & -1 \\ 0 & 0 & 1 & -1-\lambda \end{vmatrix}$$

Using cofactor expansion down the first column we see

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 & -1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{vmatrix}.$$

We have $\begin{vmatrix} -\lambda & 0 & -1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = \lambda^2(-1-\lambda) - 1 - \lambda = -\lambda^3 - \lambda^2 - \lambda - 1$

Also $\begin{vmatrix} 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{vmatrix} = 1$ since the matrix is upper triangular

Thus $\det(A - \lambda I) = -\lambda(-\lambda^3 - \lambda^2 - \lambda - 1) + 1$

$$= \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1.$$

Now we find the roots of the characteristic polynomial. Observe

$$(\lambda-1)\det(A-\lambda I) = (\lambda-1)(\lambda^4+\lambda^3+\lambda^2+\lambda+1)$$
$$= \lambda^5 - 1.$$

Since 1 is not a root of $\det(A-\lambda I)$ this means that the roots are the same as the roots of $\lambda^5 - 1$ but with 1 removed. From what we learned about complex numbers the roots have the form

$$\lambda_j = \cos\theta_j + i\sin\theta_j \quad \text{where} \quad \theta_j = \frac{2\pi j}{5} \quad \text{and } j \in \{1,2,3,4\}.$$

These are the eigenvalues of A.

Now we will find corresponding eigenvectors.

The eigenvector $\vec{x}_j$ satisfies $(A-\lambda_j I)\vec{x}_j = \vec{0}$.

We will find $\vec{x}_j$ by row reducing $A - \lambda_j I$.

$$A-\lambda_j I = \begin{pmatrix} -\lambda_j & 0 & 0 & -1 \\ 1 & -\lambda_j & 0 & -1 \\ 0 & 1 & -\lambda_j & -1 \\ 0 & 0 & 1 & -1-\lambda_j \end{pmatrix} \sim \begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ -\lambda_j & 0 & 0 & -1 \\ 0 & 1 & -\lambda_j & -1 \\ 0 & 0 & 1 & -1-\lambda_j \end{pmatrix}$$

Adding λ_j times row 1 to row 2 yields

$$\begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ -\lambda_j & 0 & 0 & -1 \\ 0 & 1 & -\lambda_j & -1 \\ 0 & 0 & 1 & -1-\lambda_j \end{pmatrix} \sim \begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ 0 & -\lambda_j^2 & 0 & -1-\lambda_j \\ 0 & 1 & -\lambda_j & -1 \\ 0 & 0 & 1 & -1-\lambda_j \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ 0 & 1 & -\lambda_j & -1 \\ 0 & -\lambda_j^2 & 0 & -1-\lambda_j \\ 0 & 0 & 1 & -1-\lambda_j \end{pmatrix} \sim \begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ 0 & 1 & -\lambda_j & -1 \\ 0 & 0 & -\lambda_j^3 & -1-\lambda_j-\lambda_j^2 \\ 0 & 0 & 1 & -1-\lambda_j \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ 0 & 1 & -\lambda_j & -1 \\ 0 & 0 & 1 & -1-\lambda_j \\ 0 & 0 & -\lambda_j^3 & -1-\lambda_j-\lambda_j^2 \end{pmatrix} \sim \begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ 0 & 1 & -\lambda_j & -1 \\ 0 & 0 & 1 & -1-\lambda_j \\ 0 & 0 & 0 & -1-\lambda_j-\lambda_j^2-\lambda_j^3-\lambda_j^4 \end{pmatrix}$$

Since the bottom right entry is the negation of the characteristic polynomial and λ_j is a root, this entry is zero. So

$$A-\lambda_j I \sim \begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ 0 & 1 & -\lambda_j & -1 \\ 0 & 0 & 1 & -1-\lambda_j \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -\lambda_j & 0 & -1 \\ 0 & 1 & 0 & -1-\lambda_j-\lambda_j^2 \\ 0 & 0 & 1 & -1-\lambda_j \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1-\lambda_j-\lambda_j^2-\lambda_j^3 \\ 0 & 1 & 0 & -1-\lambda_j-\lambda_j^2 \\ 0 & 0 & 1 & -1-\lambda_j \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have showed

$$A - \lambda_j I \sim \begin{pmatrix} 1 & 0 & 0 & -1-\lambda_j-\lambda_j^2-\lambda_j^3 \\ 0 & 1 & 0 & -1-\lambda_j-\lambda_j^2 \\ 0 & 0 & 1 & -1-\lambda_j \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so the corresponding eigenvector is

$$\vec{x}_j = \begin{pmatrix} 1+\lambda_j+\lambda_j^2+\lambda_j^3 \\ 1+\lambda_j+\lambda_j^2 \\ 1+\lambda_j \\ 1 \end{pmatrix}.$$

$-1 = \lambda_j + \lambda_j^2 + \lambda_j^3 + \lambda_j^4.$

Check: $A\vec{x}_j = \begin{pmatrix} -1 \\ \lambda_j+\lambda_j^2+\lambda_j^3 \\ \lambda_j+\lambda_j^2 \\ \lambda_j \end{pmatrix}$

We have $A = X \Lambda X^{-1}$ where

$X = \left(\vec{x}_1 \quad \vec{x}_2 \quad \vec{x}_3 \quad \vec{x}_4 \right)$ and

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$