THE BASIS FOR THE ROW SPACE

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Theorem. Let A be a matrix and R be its reduced echelon form. The non-zero rows of R form a basis for the row space of A, $C(A^T)$.

More generally, this is true if R is just taken to be an echelon form of the matrix, but the direct proof would be slightly more subtle in this case.

Rather than a proof, we will give an example demonstrating why the statement is true.

Consider the case of

$$A = \begin{pmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & 4 & 2 & 7 \\ 0 & 0 & 1 & -2 & -3 \end{pmatrix}.$$

If we apply elimination to obtain the reduced echelon form R we see that

$$A = \begin{pmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & 4 & 2 & 7 \\ 0 & 0 & 1 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -7 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

The theorem then says that a basis for the row space $C(A^T)$ is given by the non-zero rows of R, i.e.,

$$\mathcal{B} = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\} = \left\{ \begin{pmatrix} 1 \\ -7 \\ 0 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \right\}.$$

Why is $\mathcal{B}$ a basis for $C(A^T)$? We must check that these $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ are linearly independent and span the row space.

Linear Independence. We need to show that the only solution for c_1, c_2 and c_3 in

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = \mathbf{0}$$

occurs when $c_1 = c_2 = c_3 = 0$. This is true because we can simplify the left hand side

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = c_1 \begin{pmatrix} 1 \\ -7 \\ 0 \\ 0 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} c_1 \\ -7c_1 \\ c_2 \\ c_3 \\ -4c_1 + \frac{3}{2}c_3 \end{pmatrix}.$$

Observe that three of the entries of this vector are just c_1, c_2 and c_3, and they appear in the positions of the pivot entries in the rows of R. (This statement is true for any matrix A and reduced matrix R.) So in particular, we see that $c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = 0$ implies $c_1 = c_2 = c_3 = 0$ as required for verifying that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are linearly independent.

Spanning the row space. We need to show that $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ span the row space. This means that for any vector $\mathbf{v}$ in the row space $C(A^T)$, there should be scalars c_1, c_2 and c_3 so that

$$\mathbf{v} = c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3. \tag{1}$$

We want to convert this into a matrix equation. Since the transpose operation switches rows and columns, the elements the row space is given by $A^T\mathbf{x}$ for some $\mathbf{x}$. So we can take $\mathbf{v} = A\mathbf{x}$ for some $\mathbf{x}$. Similarly,

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = R^T \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix},$$

where c_4 can be anything since the 4th row of R is zero (and hence the 4th column of R^T is zero). Thus solving equation (1) is the same as solving the matrix equation

$$A^T\mathbf{x} = R^T\mathbf{c} \quad \text{for } \mathbf{c} \text{ where } \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

Now consider the relationship between A and its reduced echelon form R: We know there is an invertible matrix E so that $R = EA$. Thus we also have $A = E^{-1}R$. Taking the transpose of this equation, we see $A^T = R^T(E^{-1})^T$. Substituting, we see that we are looking for the solution to

$$R^T(E^{-1})^T\mathbf{x} = R^T\mathbf{c}.$$

Well, this equation has an obvious solution, namely we take $\mathbf{c} = (E^{-1})^T\mathbf{x}$. This choice of $\mathbf{c}$ gives us our entries c_1, c_2, and c_3 satisfying equation 1.