THE BASIS FOR THE COLUMN SPACE

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Theorem. Let A be a matrix. The pivot columns of the original matrix A form a basis for the column space $C(A)$.

Rather than a proof, we will give an example demonstrating why the statement is true.

Consider the case of

$$A = \begin{pmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & 4 & 2 & 7 \\ 0 & 0 & 1 & -2 & -3 \end{pmatrix}.$$

If we apply elimination to obtain the reduced echelon form R we see that

$$A = \begin{pmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & 4 & 2 & 7 \\ 0 & 0 & 1 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -7 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

There are pivots in columns 1, 3 and 4. This means that the theorem above promises that the associated columns of A are a basis for $C(A)$:

$$\mathcal{B} = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 16 \\ -2 \end{pmatrix} \right\}.$$

Why is $\mathcal{B}$ a basis? We must check that these columns are independent and span the column space.

The most basic observation from elimination is that there is an invertible matrix E so that $R = EA$. This also means that $A = E^{-1}R$. We use this in both parts of the proof.

Linear Independence. First let us check that the pivot columns $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ of A are linearly independent. By definition, we need to show that the only solution to

$$c_1\mathbf{a}_1 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4 = \mathbf{0}$$

is given by $c_1 = c_3 = c_4 = 0$. We can write the equation in a matrix form as

$$A \begin{pmatrix} c_1 \\ 0 \\ c_3 \\ c_4 \\ 0 \end{pmatrix} = \mathbf{0}.$$

Since E is invertible, the solution set is the same as the solution set obtained when you left multiply by E. Since $EA = R$ and $E\mathbf{0} = \mathbf{0}$, we see that this equation is the same as:

$$R\begin{pmatrix} c_1 \\ 0 \\ c_3 \\ c_4 \\ 0 \end{pmatrix} = \mathbf{0}.$$

Now recall that a matrix times a vector gives a linear combination of the columns. The columns associated to c_1, c_3 and c_4 are pivot columns so they have exactly one 1 (in the pivot position) and all other entries are zero. Moreover, the pivots all appear in different rows. In this case have

$$R\begin{pmatrix} c_1 \\ 0 \\ c_3 \\ c_4 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_3 \\ c_4 \\ 0 \end{pmatrix}.$$

Thus the only way this vector can equal the zero vector is if $c_1 = c_3 = c_4 = 0$ as claimed.

Spans. Now we must check that the vectors $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ span the column space $C(A)$. A vector in $C(A)$ always has the form $A\mathbf{x}$ for some vector $\mathbf{x}$ (again since a matrix times a vector is a linear combination of the columns). So, we need to show that for any $\mathbf{x}$, there are scalars c_1, c_3 and c_4 so that

$$A\mathbf{x} = c_1\mathbf{a}_1 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4.$$

As in the previous part of the proof, we can rewrite the right hand side as a matrix times a vector, so this is equivalent to finding c_1, c_3 and c_4 satisfying

$$A\mathbf{x} = A\begin{pmatrix} c_1 \\ 0 \\ c_3 \\ c_4 \\ 0 \end{pmatrix}.$$

Again left multiplying both sides by E will produce an equation with the same solution set since E is invertible. (We are solving for c_1, c_3 and c_4.) Again $EA = R$ so this is equivalent to solving

$$R\mathbf{x} = R\begin{pmatrix} c_1 \\ 0 \\ c_3 \\ c_4 \\ 0 \end{pmatrix}.$$

Since R is in reduced echelon form, we can simplify the right hand side obtaining

$$R\mathbf{x} = \begin{pmatrix} c_1 \\ c_3 \\ c_4 \\ 0 \end{pmatrix}.$$

Now notice that we can always solve this equation for c_1, c_3 and c_4 as long as the last coordinate of $R\mathbf{x}$ is zero. Also notice that the last coordinate of $R\mathbf{x}$ is definitely zero since the last row of R is all zeros. Thus, we can always find c_1, c_3 and c_4 solving this equation as required by the definition of span.