FUNCTION INVERSES AND MATRIX INVERSES

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Definition: Let A and B be sets. Functions $f : A \to B$ and $g : B \to A$ are *inverses* if $f \circ g$ is the identity map on A and $g \circ f$ is the identity map on B. The function g is the *inverse* of f and is usually denoted g^{-1}.

Calculus example: The function $f(x) = e^x$ and $g(x) = \ln x$ are inverses if we consider the domain and co-domain to be given as $f : \mathbb{R} \to (0, \infty)$ and $g : (0, \infty) \to \mathbb{R}$.

For Linear Transformations: The definition above applies to linear transformations since linear transformations are functions. Let m and n be positive integers. Linear transformations $S : \mathbb{R}^m \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are *inverses* if $S \circ T$ and $T \circ S$ are identity maps.

Recall: Linear transformations in this context are the same as matrices. So, if $S : \mathbb{R}^m \to \mathbb{R}^n$ is linear there is an $n \times m$ matrix A so that $S(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^m$. Similarly, there is a $m \times n$ matrix B so that $T(\mathbf{w}) = B\mathbf{v}$ for all $\mathbf{w} \in \mathbb{R}^n$. If they are inverses then $S \circ T = Id_n$ and $T \circ S = Id_m$ where $Id_m : \mathbb{R}^m \to \mathbb{R}^m$ and $Id_n : \mathbb{R}^n \to \mathbb{R}^n$ are identity maps. Identity maps correspond to identity matrices. If I_m is the $m \times m$ identity matrix and I_n is the $n \times n$ identity matrix then $Id_m(\mathbf{v}) = I_m\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^m$ and $Id_n(\mathbf{w}) = I_n\mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}^n$.

Exercise: Use that $S(\mathbf{v}) = A\mathbf{v}$ and $Id_m(\mathbf{v}) = I_m(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^m$ and that $T(\mathbf{w}) = B\mathbf{w}$ and $Id_n(\mathbf{w}) = I_n(\mathbf{w})$ for all $\mathbf{w} \in \mathbb{R}^n$ to convert the following statement about linear transformations to one only involving matrices:

$$T \circ S(\mathbf{v}) = Id_m(\mathbf{v}) \text{ for all } \mathbf{v} \in \mathbb{R}^m \quad \text{and} \quad S \circ T(\mathbf{w}) = Id_m(\mathbf{w}) \text{ for all } \mathbf{w} \in \mathbb{R}^n.$$

Definition: Suppose A is an $n \times m$ matrix and B is a an $m \times n$ matrix. Then A and B are *inverses* if AB and BA are identity matrices. We call B the *inverse* of A and typically denote this inverse by A^{-1}. We call a matrix *invertible* if it has an inverse.

Example: Show that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc \neq 0$ then it has an inverse given by

$$B = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example: To what extent are the matrices below inverses of one another?

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Remark: In the above example, B is called a *right-inverse* of A. Also A is called a *left-inverse of* B. They are not true inverses!

Remark: In order for A and B to be inverses of each other, they have to be square and of the same dimensions. So there must be an n so that both A and B are $n \times n$.

Proposition: Suppose we are given an $m \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^n$. Suppose E is an invertible $n \times n$ matrix. Let $U = EA$ and $\mathbf{c} = E\mathbf{b}$. Then the solution sets to the equations $A\mathbf{x} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$ are the same.

Proof: