§8.1: LINEAR TRANSFORMATIONS

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Definition (§8.1, page 401) A *transformation* T assigns an output $T(\mathbf{v})$ to each input vector in its domain $\mathbf{V}$. The transformation is *linear* if it meets these requirements for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{V}$:

$$\text{(a) } T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) \qquad \text{(b)} T(c\mathbf{v}) = cT(\mathbf{v}) \text{ for all scalars } c.$$

The following fact is very important. You should be able to prove it!

Proposition: The composition of two linear transformations is a linear transformation.

We remark that the composition of two transformations (or maps) S and T is defined when the outputs of S lie in the domain of T. The *composition* is the transformation denoted $T \circ S$ and defined to have the same domain as S and for every $\mathbf{v}$ in this domain,

$$T \circ S(\mathbf{v}) = T\big(S(\mathbf{v})\big).$$

Proof:

If $T : R^n \to \mathbb{R}^m$ is a linear transformation its *matrix* is the $m \times n$ matrix whose i-th column is $T(\mathbf{e}_i)$ where $\mathbf{e}_i \in \mathbb{R}^n$ is the vector whose i-th entry is one and whose other entries are all zeros.

If A is the matrix of the linear transformation $T : R^n \to \mathbb{R}^m$ then $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

Important: The linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ are exactly the $m \times n$ matrices! If you have such a linear transformation T you can find it's matrix. Conversely, if you have an $m \times n$ matrix you can define the linear transformation $T(\mathbf{v}) = A\mathbf{v}$.

Example: The identity map $Id : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $Id(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$. What is the matrix of Id? This is the 3×3 *identity matrix* and is usually denoted I. What about for dimensions other than 3?

Composition and Matrix Multiplication: Suppose $S(\mathbf{v}) = A\mathbf{v}$ and $T(\mathbf{w}) = B\mathbf{w}$ are linear transformations with there associated matrices. When is the composition $T \circ S$ well defined?

Assume $T \circ S$ is well defined. What is the matrix of the transformation $T \circ S$? You can describe it in terms of the rows of B and the columns of A...

We've discovered matrix multiplication! The map $T \circ S(\mathbf{v}) = B(A\mathbf{v})$. We define the matrix product BA to be the matrix of the transformation $T \circ S(\mathbf{v})$.