Math 308: Bridge to Advanced Math
Notes on Right and Left Inverses

Throughout this note let A and B be sets with $A \neq \emptyset$ and let f be a function from A to B.

Theorem 1. A function $f : A \to B$ has a right inverse if and only if f is surjective.

The proof needs two important concepts. First there is the following axiom of set theory:

Axiom of Choice. Let I be a set. Recall that family of sets indexed by I is a collection of sets of the form $\{S_i\}_{i\in I}$. This means that we have chosen a set S_i for every $i \in I$. Informally, the Axiom of Choice tells us that we can simultaneously choose an element of S_i for each $i \in I$ so long as each $i \in S_i$ is nonempty. Formally, the *Axiom of Choice* says that if $\{S_i\}_{i\in I}$ is an indexed collection of sets and each S_i is non-empty, then there is a *choice function* $c : I \to \bigcup_{i\in I} S_i$ so that $c(i) \in S_i$ for each $i \in I$.

Definition: If $f : A \to B$ is a function and $C \subset B$, then the *preimage* of C under f is

$$f^{-1}(C) = \{a \in A : \ f(a) \in C\}.$$

Proof. First suppose that f has a right inverse $g : B \to A$. We will verify that f is surjective using the definition. We will show that for every $b \in B$, there is an $a \in A$ so that $f(a) = b$. Let $b \in B$ be arbitrary. Then define $a = g(b)$. Since g is a right inverse,

$$f(a) = f \circ g(b) = b.$$

This verifies the definition.

Now suppose that f is surjective. Then for every $b \in B$, there is at least one $a \in A$ so that $f(a) = b$. Thus, for each $b \in B$, the preimage $f^{-1}(\{b\})$ is non-empty. Therefore $\big\{\{f^{-1}(\{b\})\}\big\}_{b\in B}$ is a family of non-empty sets indexed by elements of the set B. Observe that every $a \in A$ is in the preimage $f^{-1}\big(\{f(a)\}\big)$, so $\bigcup_{b\in B} f^{-1}(\{b\}) = A$. The Axiom of Choice allows us to choose a choice function $c : B \to A$ so that $c(b) \in f^{-1}(\{b\})$ for all $b \in B$. By definition of the preimage, this means that $f \circ c(b)\ in\{b\}$ or equivalently $f \circ c(b) = b$ for all $b \in B$. Therefore, $c : B \to A$ is a right inverse of f. □

Theorem 2. Suppose $A \neq \emptyset$. A function $f : A \to B$ has a left inverse if and only if f is injective.

Remark: We need $A \neq \emptyset$ because any function from the empty set to a non-empty set is injective, but there are no functions from a non-empty set to an empty set!

Proof. First suppose that f has a left inverse $g : B \to A$. We will show f is injective by verifying the definition. Let $a_1, a_2 \in A$ and suppose that $f(a_1) = f(a_2)$. We must show that $a_1 = a_2$. Since $f(a_1) = f(a_2)$, we know that $g \circ f(a_1) = g \circ f(a_2)$. Then since g is a left inverse,

$$g \circ f(a_1) = a_1 \quad \text{and} \quad g \circ f(a_2) = a_2.$$

Therefore $a_1 = a_2$, which proves that f is injective.

Now suppose f is injective. Then for every $b \in B$, there is at most one $a \in A$ so that $f(a) = b$. Recall that the range of f is $f(A) = \{f(a) \in B : \ a \in A\}$, i.e., the range is the set $f(A)$ of elements $b \in B$ for which there is an $a \in A$ with $f(a) = b$. Now define an alternate version of f with a restricted codomain. Define $f' : A \to f(A)$ by $f'(a) = f(a)$. Then by definition of the range, f'

is surjective. Therefore, by Theorem 1, there is a right-inverse of f' which we call $g : f(A) \to A$. Since A is non-empty, we can choose an $a_0 \in A$. Define $h : B \to A$ by

$$h(b) = \begin{cases} g(b) & \text{if } b \in f(A), \\ a_0 & \text{if } b \in B \smallsetminus f(A). \end{cases}$$

We will check that h is a left inverse. Let $a \in A$. We must show that $h \circ f(a) = a$ for all $a \in A$. Fix an $a \in A$. Then $f(a) \in f(A)$ by definition of the image. So, $h \circ f(a) = g \circ f(a)$. By post-composing with f' and using that g is a right-inverse of f', we see that

$$f' \circ h \circ f(a) = f' \circ g \circ f(a) = f(a).$$

Recalling that $f' = f$, we see that

$$f \circ h \circ f(a) = f(a).$$

Now because f is injective, it must be that $h \circ f(a) = a$. This shows that h is a left inverse. $\square$