Math 70100: Functions of a Real Variable I: Midterm Solutions

Friday, October 24th, 2014

Prof. Hooper

1. Complete the following definitions:
   (a) (8 points) Let X be a metric space, and let $A \subset X$. The *boundary of A* is...

   Solution: the set of points $x \in X$ so that every open set containing x intersects both A and $X \setminus A$.

   (b) (8 points) A metric space X is *disconnected* if ...

   Solution: there are non-empty disjoint open subsets $U, V \subset X$ so that $X = U \cup V$.

2. (12 points) State one of the forms of the Baire Category theorem for complete metric spaces.

Solution:

Solution 1. In a complete metric space, the countable intersection of open dense sets is dense.

Solution 2. If a complete metric space is a countable union of closed sets, then one of the closed sets has non-empty interior.

3. (15 points) Let $\{x_n\}$ be a sequence in a metric space X. Show that the set of limit points of convergent subsequences of $\{x_n\}$ is a closed subset of X.

Solution:

Solution 1: Let S denote the set of limit points of convergent subsequences of $\{x_n\}$.

Let $\{y^m\}$ be a sequence in S which converges to a point $y \in X$. It suffices to prove that $y \in S$. Since $y^m \in S$, for each m, there is an increasing sequence of integers $j \mapsto n(m,j)$ so that the subsequence $\{x_{n(m,j)}\}$ converges to y^m.

We will inductively define an increasing sequence of integers $m \mapsto p(m)$ so that $\{x_{p(m)}\}$ tends to y as $m \to \infty$, thus verifying that $y \in S$. We begin with the base case of defining $p(1)$. Since $x_{n(1,j)} \to y^1$, there is a J_1 so that

$$j > J_1 \quad \text{implies} \quad d(x_{n(1,j)}, y^1) < \frac{1}{2}.$$

We set $p(1) = n(1,j)$ for some $j > J_1$. Now inductively suppose $p(m)$ is defined. We will define $p(m+1)$. Since $x_{n(m+1,j)} \to y^{m+1}$, there is a J_{m+1} so that

$$j > J_{m+1} \quad \text{implies} \quad d(x_{n(m+1,j)}, y^{m+1}) < \frac{1}{2^{m+1}}.$$

We set $p(m+1) = n(m+1, j)$ for some $j > \max\big(J_{m+1}, p(m)\big)$.

Observe that the inductive argument produces a subsequence $\{x_{p(m)}\}$ satisfying

$$d(x_{p(m)}, y^m) < \frac{1}{2^m} \quad \text{for all } m \in \mathbb{N}.$$

We claim that this implies that $\{x_{p(m)}\}$ tends to y. By the triangle inequality,

$$0 \leq d(x_{p(m)}, y) \leq d(x_{p(m)}, y^m) + d(y^m, y) < \frac{1}{2^m} + d(y^m, y).$$

As $m \to \infty$ both $\frac{1}{2^m} \to 0$ and $d(y^m, y) \to 0$. (The later is because $y^m \to y$.) Thus, $d(x_{p(m)}, y) \to 0$ as $m \to \infty$. That is, $\{x_{p(m)}\}$ converges to y.

Solution 2: Fix $\{x_n\}$. Let S denote the set of limit points of convergent subsequences of $\{x_n\}$. We will show that $X \smallsetminus S$ is open.

Observation. Observe that there is a subsequence of $\{x_n\}$ which converges to x if and only if for any $\epsilon > 0$, there are infinitely many n so that $d(x_n, x) < \epsilon$.

Remark: The observation above does not require proof, because it is fairly clear. But, I will prove this observation is correct for completeness.

Proof of Observation. First suppose $\{x_{n_k}\}$ is a subsequence converging to $x \in X$. Then for $\epsilon > 0$, there is K so that $k > K$ implies $d(x_{n_k}, x) < \epsilon$. The set of all n_k with $k > K$ is infinite and satisfies the statement. Conversely, suppose $x \in X$ and suppose that for any $\epsilon > 0$ there is an infinite set $N_\epsilon \subset \mathbb{N}$ so that $n \in N_\epsilon$ implies $d(x, x_n) < \epsilon$. Then we can define a subsequence converging to x inductively. Choose any $n_1 \in N_1$. Now suppose n_k is defined for some integer $k \geq 1$. Then because $N_{1/(k+1)}$ is infinite, we can choose an $n_{k+1} \in N_{1/(k+1)}$ so that $n_{k+1} > n_k$. We see that the inductively defined subsequence x_{n_k} converges to x because

$$d(x_{n_k}, x) < \frac{1}{k} \quad \text{so} \quad \lim_{k \to \infty} d(x_{n_k}, x) = 0.$$

Now we will show that $X \smallsetminus S$ is open. Since the empty set is open, we will assume $X \smallsetminus S \neq \emptyset$. Let $x \in X \smallsetminus S$. Then by our observation, there is an $\epsilon > 0$ so that

$$N = \{n \in \mathbb{N} \ : \ x_n \in B_\epsilon(x)\} \quad \text{is finite,}$$

where $B_\epsilon(x)$ denotes the open ball of radius ϵ about x. We claim that $B_\epsilon(x) \subset X \smallsetminus S$. Because $x \in X \smallsetminus S$ was arbitrary, this implies that $X \smallsetminus S$ is open. Let $y \in B_\epsilon(x)$, and let $r = \epsilon - d(x, y) > 0$. The triangle inequality implies that $B_r(y) \subset B_\epsilon(x)$. So the set

$$M = \{n \in \mathbb{N} \ : \ x_n \in B_r(y)\} \quad \text{is a subset of } N.$$

Then because N is finite, it must be true that M is finite. So again by our observation above, we see that y is not a limit of a convergent subsequence of $\{x_n\}$. This completes the proof that $B_\epsilon(x) \subset X \smallsetminus S$ as needed to show that $X \smallsetminus S$ is open.

4. (15 points) A topological space is *separable* if it has a countable dense subset. A subspace is *second*

countable if it has a countable base. Recall that a metric space is separable if and only if it is second countable.

Show that any totally bounded metric space is separable. (*Remark:* If you can not recall the definition of totally bounded, then prove that every compact metric space is separable for a loss of a few points.)

Solution: Let X be a totally bounded metric space. Then for each $n \in \mathbb{N}$, there is a finite collection of points A_n so that

$$X = \bigcup_{a \in A_n} B_{\frac{1}{n}}(a). \tag{1}$$

Let A be the countable set $\bigcup_{n \in \mathbb{N}} A_n$.

We claim that A is dense in X, which gives us that X is separable. Suppose to the contrary that A is not dense. Then there is an open ball $B_\epsilon(y)$, with $\epsilon > 0$ and $y \in X$, which does not intersect A. Choose $n \in \mathbb{N}$ so that $\frac{1}{n} < \epsilon$. Then because of equation 1, we can find an $a \in A_n \subset A$ so that $y \in B_{\frac{1}{n}}(a)$. Then $d(a, y) < \frac{1}{n} < \epsilon$. We conclude that $a \in B_\epsilon(y)$, which contradicts the statement that $B_\epsilon(y)$ does not intersect A.

5. (15 points) Let X be a compact metric space and let $x_0 \in X$ be a basepoint. Let $C(X)$ denote the collection of continuous functions from X to $\mathbb{R}$ with the uniform norm. A function $f : X \to \mathbb{R}$ is said to be K-*Lipschitz* for some $K > 0$ if

$$|f(x) - f(y)| \leq Kd(x, y) \quad \text{for each } x, y, \in X.$$

Fix some $K > 0$. Show that the collection of functions

$$\mathcal{L} = \{f \in C(X) \ : \ f \text{ is } K\text{-Lipschitz and } f(x_0) = 0\}$$

is a compact subset of $C(X)$.

Solution: We will first show that $\mathcal{L}$ has compact closure in $C(X)$ using the Arzelá-Ascoli theorem. In order to do this we must verify that $\mathcal{L}$ is equicontinuous and pointwise totally bounded.

We begin by checking that $\mathcal{L}$ is pointwise totally bounded. Fix some $x \in X$. Then observe because the functions in $\mathcal{L}$ are K-Lipschitz and send x_0 to zero, we have

$$|f(x)| = |f(x) - f(x_0)| \leq Kd(x, y) \quad \text{for all } f \in \mathcal{L}.$$

The quantity $Kd(x, y)$ therefore serves as an upper bound for $\{|f(x)| \ : \ f \in \mathcal{L}\}$. Therefore, $\mathcal{L}$ is pointwise bounded. In $\mathbb{R}$, bounded implies totally bounded, so $\mathcal{L}$ is pointwise totally bounded.

To see $\mathcal{L}$ is equicontinuous fix $x \in X$ and $\epsilon > 0$. Then if $y \in X$ and $d(x, y) < \frac{\epsilon}{K}$, we have

$$|f(x) - f(y)| \leq Kd(x, y) < \epsilon \quad \text{for all } f \in \mathcal{L},$$

because our functions are K-Lipschitz. This verifies the equicontinuity of $\mathcal{L}$.

We have shown $\mathcal{L}$ is equicontinuous and pointwise totally bounded, so by the Arzelá-Ascoli theorem, $\mathcal{L}$ has compact closure. Therefore, to show $\mathcal{L}$ is compact, it suffices to verify that it is closed.

To see $\mathcal{L}$ is closed, let $\{f_n\}$ be a sequence in $\mathcal{L}$ which converges to $f \in C(X)$. We claim that $f \in \mathcal{L}$. There are two conditions to check based on the definition of $\mathcal{L}$. First, to see f is K-Lipschitz, pick $x, y \in X$. Then,

$$|f(x) - f(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)|.$$

Since each f_n is K-Lipschitz, we have $|f_n(x) - f_n(y)| \leq Kd(x, y)$ for each $n \in \mathbb{N}$. The limiting value mist then also satisfy this inequality, so f is K-Lipschitz as well. Second, we need to observe that $f(x_0) = 0$. Again,

$$f(x_0) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} 0 = 0.$$

We have shown that $f \in \mathcal{L}$ as claimed.

6. (a) (12 points) State the Stone-Weierstrass Theorem. (It suffices to state a version for functions from a compact metric space into $\mathbb{R}$.)

Solution: Let X be a compact metric space. Consider the space $C(X)$ of continuous functions from X into $\mathbb{R}$ with the uniform norm. Let $\mathcal{A} \subset C(X)$ be an algebra of functions which vanishes nowhere and separates points. Then $\mathcal{A}$ is dense in $C(X)$.

(b) (15 points) Show that $\lim_{n\to\infty} \int_0^1 nx^n f(x)\, dx = f(1)$ for any continuous function $f : [0, 1] \to \mathbb{R}$.

Solution: We begin by showing that this is true when $f(x) = x^m$ for some integer $m \geq 0$. Observe that

$$\lim_{n\to\infty} \int_0^1 nx^n(x^m)\, dx = \lim_{n\to\infty} \left[\frac{n}{n+m} x^{n+m+1}\right]_0^1 dx = \lim_{n\to\infty} \frac{n}{n+m} = 1. \tag{2}$$

Since here $f(x) = x^m$ we have $f(1) = 1$, and we have verified the result in this case.

We will now show that the statement is true for any polynomial $p(x) = \sum_{i=0}^m a_i x^i$. Observe by linearity of the integral,

$$\int_0^1 nx^n p(x)\, dx = \sum_{i=0}^m a_i \int_0^1 nx^n x^i\, dx.$$

Therefore by passing the limit into the sum and using our calculation in equation 2 to evaluate the limits, we see

$$\lim_{n\to\infty} \int_0^1 nx^n p(x)\, dx = \sum_{i=0}^m a_i \lim_{n\to\infty} \int_0^1 nx^n x^i\, dx = \sum_{i=0}^m a_i = p(1).$$

Now let $f : [0, 1] \to \mathbb{R}$ be an arbitrary continuous function. For each integer $n \geq 1$, let

$$I_n = \int_0^1 nx^n f(x)\, dx.$$

We claim that $I_n \to f(1)$. To verify this, choose some $\epsilon > 0$. It suffices to verify that there is an N so that $n > N$ implies that

$$|I_n - f(1)| < \epsilon.$$

Recall that by Weierstrass' theorem, the polynomials are uniformly dense in the space of continuous functions from $[0,1]$ to $\mathbb{R}$. Therefore, for any $\epsilon > 0$, we can find a polynomial p so $\|f-p\| < \frac{\epsilon}{4}$, where $\|\cdot\|$ is the uniform norm. So, we have

$$\frac{-\epsilon}{4} < f(x) - p(x) < \frac{\epsilon}{4} \quad \text{for every } x \in [0,1].$$

Multiplying through by nx^n which is positive when $x > 0$, we see:

$$\frac{-\epsilon n x^n}{4} \leq nx^n f(x) - nx^n p(x) \leq \frac{\epsilon n x^n}{4} \quad \text{for every } x \in [0,1].$$

Observe that the integral of nx^n over $[0,1]$ is $\frac{n}{n+1} < 1$. So, we see

$$\frac{-\epsilon}{4} < \frac{-\epsilon n}{4(n+1)} \leq \int_0^1 nx^n f(x)\ dx - \int_0^1 nx^n p(x)\ dx \leq \frac{\epsilon n}{4(n+1)} < \frac{\epsilon}{4}.$$

In summary, we have

$$\left| I_n - \int_0^1 nx^n p(x)\ dx \right| < \frac{\epsilon}{4}.$$

Now recall that we proved that

$$\lim_{n\to\infty} \int_0^1 nx^n p(x)\ dx = p(1).$$

So, we can choose an N so that $n > N$ implies

$$\left| \int_0^1 nx^n p(x)\ dx - p(1) \right| < \frac{\epsilon}{2}.$$

The final ingredient is the observation that $\|f-p\| < \frac{\epsilon}{4}$ implies that $|f(1) - p(1)| < \frac{\epsilon}{4}$. So, by the triangle inequality, for each $n > N$, we have

$$|I_n - f(1)| \leq \left| I_n - \int nx^n p(x)\ dx \right| + \left| \int nx^n p(x)\ dx - p(1) \right| + |p(1) - f(1)| < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon.$$