METRIC COMPLETIONS

W. PATRICK HOOPER

In this note, we continue our discussion from class on metric completions. I will attempt to give a careful and complete exposition of the proof that every metric space has a completion, and that this completion is unique up to isometry. Existence is the difficult proof. I will give a summary of a short "high-level" proof, and also complete the more conceptually natural proof begun in class using equivalence classes of Cauchy sequences to complete a metric space.

1. Statements of results

We begin by recalling a few definitions. Let X and Y be metric spaces. A *Cauchy sequence* in X is a sequence $\{x_n \in X\}_{n\in\mathbb{N}}$ so that for every $\epsilon > 0$, there is an N such that $n, m > N$ implies that $d(x_n, x_m) < \epsilon$. Recall that only Cauchy sequences converge. The metric space X is said to be *complete* if every Cauchy sequence in X converges. An *isometry* $\phi : X \to Y$ is a map so that

$$d_Y\big(\phi(x), \phi(y)\big) = d_X(x, y).$$

(*Remark*: I believe that in some contexts it is assumed that an isometry is a bijection. The isometries in this document are not assumed to be surjective, but the definition implies injectivity.)

We say that a *metric completion* of a metric space (X, d) is a complete metric space $(\tilde{X}, \tilde{d})$ together with an isometry $\phi : X \to \tilde{X}$ so that $\phi(X)$ is dense inside of $\tilde{X}$. The main goal of this note is to prove that this definition is not vacuous:

Theorem 1 (Existence of metric completions)**.** *Every metric space has a metric completion.*

We will give two proofs of this theorem later in this document. But first we observe that if metric completions exist, they are (in a metric sense) unique. Concretely:

Proposition 2 (Uniqueness of metric completions)**.** *Let (X, d) be a metric space. Let $(\tilde{X}, \tilde{d})$ and $(\hat{X}, \hat{d})$ be two complete metric spaces, and let $\tilde{\phi} : X \to \tilde{X}$ and $\hat{\phi} : X \to \hat{X}$ be isometries with dense images. (So, both are metric completions.) Then, there is a unique isometry $\psi : \tilde{X} \to \hat{X}$ so that $\hat{\phi}(x) = \psi \circ \tilde{\phi}(x)$ for all $x \in X$.*

We will prove the uniqueness proposition first, and then give two proofs of the existence theorem.

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2. Proof of the uniqueness of metric completions

Proof of Proposition 2. We will give a direct proof. Assume all the notation from the proposition.

We must define $\psi : \tilde{X} \to \hat{X}$. For each $\tilde{x} \in \tilde{X}$, by density of $\tilde{\phi}(X)$ within $\tilde{X}$, we can make a choice of a sequence $\alpha(\tilde{x}) = \{x_n \in X\}_{n \in \mathbb{N}}$ so that $\lim_{n\to\infty} \tilde{\phi}(x_n) = \tilde{x}$. Fix $\tilde{x}$ and let $\{x_n\} = \alpha(\tilde{x})$. Observe that $\{\tilde{\phi}(x_n)\}$ is Cauchy because it converges. Then because $\tilde{\phi}$ is an isometry, it follows that $\{x_n\}$ is Cauchy in X. Similarly, because $\hat{\phi}$ is an isometry, it follows that $\{\hat{\phi}(x_n)\}$ is Cauchy in $\hat{X}$. Then because $\hat{X}$ is complete, the sequence $\{\hat{\phi}(x_n)\}$ has a limit, which we call $\psi(\hat{x})$. This defines a map $\psi : \tilde{X} \to \hat{X}$.

First we claim that $\hat{\phi}(x) = \psi \circ \tilde{\phi}(x)$ for all $x \in X$. Fix $x \in X$. Let $\{x_n\} = \alpha \circ \tilde{\phi}(x)$. Then $\tilde{\phi}(x_n) \to \tilde{\phi}(x)$. Since $\tilde{\phi}$ was an isometry, we see $x_n \to x$. And because $\hat{\phi}$ is an isometry, we see $\hat{\phi}(x_n) \to \hat{\phi}(x)$. By definition of ψ, we have

$$\psi \circ \tilde{\phi}(x) = \lim_{n\to\infty} \hat{\phi}(x_n) = \hat{\phi}(x),$$

as desired.

Finally, we claim that ψ is an isometry. Let $\tilde{x}, \tilde{y} \in \tilde{X}$. Let $\{x_n\} = \alpha(\tilde{x})$ and $\{y_n\} = \alpha(y_n)$. Then by definition of ψ,

$$\hat{d}\big(\psi(\tilde{x}), \psi(\tilde{y})\big) = \lim_{n\to\infty} \hat{d}\big(\hat{\phi}(x_n), \hat{\phi}(y_n)\big).$$

Now observe that because $\hat{\phi}$ and $\tilde{\phi}$ are isometries, for all n,

$$\hat{d}\big(\hat{\phi}(x_n), \hat{\phi}(y_n)\big) = d(x_n, y_n) = \tilde{d}\big(\tilde{\phi}(x_n), \tilde{\phi}(y_n)\big).$$

Then because x_n and y_n were defined so that $\tilde{\phi}(x_n) \to \tilde{x}$ and $\tilde{\phi}(y_n) \to \tilde{y}$, by continuity of the distance function, we see

$$\hat{d}\big(\psi(\tilde{x}), \psi(\tilde{y})\big) = \lim_{n\to\infty} \tilde{d}\big(\tilde{\phi}(x_n), \tilde{\phi}(y_n)\big) = \tilde{d}(\tilde{x}, \tilde{y}).$$

This shows ψ is an isometry, as claimed. □

3. An abstract proof of the existence of Metric completions

We now consider proving the theorem that metric spaces admit completions. I will give an abstract proof first, following an exercise from Lang's book.

Proof of Theorem 1. Let (X, d) be a metric space. We recall in problem (5) of our first homework assignment, which was based on Lang II.5.5c. For each $x \in X$, define the function f_x on X by $f_x(y) = d(x, y)$. In two parts, we showed:

(a) If $x, y \in X$, then $d(x, y) = \|f_x - f_y\|$, where $\|\cdot\|$ denotes the sup (or uniform) norm.
(b) Let a be a fixed element of X, and let $g_x = f_x - f_a$. Then, the map $x \mapsto g_x$ is a isometry of X into the normed space of bounded functions on X.

Let $BC(X)$ denote the real-valued bounded and continuous functions on X. In summary, $g_x \in BC(X)$ for all $x \in X$ and $x \mapsto g_x$ is an isometry.

We showed in problem (6) of the same homework, which was based on Lang's II.5.8, that the space $B(X)$ of bounded real-valued functions on X is complete. (This only required X to be a topological space and held more generally for bounded functions to a Banach space.) We also showed that the bounded continuous functions $BC(X)$ form a closed subset of $B(X)$. It follows that $BC(X)$ is complete, because any closed subset of a complete metric space is complete.

Now let $\tilde{X}$ be the uniform closure of the set

$$\{g_x \in BC(X) \ : \ x \in X\}.$$

Again, $\tilde{X}$ is complete, because it is a closed subset of a complete space. We showed the map

$$\phi : X \to \tilde{X}; \quad \phi(x) = g_x$$

is an isometry. Also, by definition of the closure, we see that $\phi(X)$ is dense inside of $\tilde{X}$. Therefore, $\tilde{X}$ is a metric completion of X. □

4. Metric completion from equivalence classes of Cauchy sequences

To my mind the proof I began discussing in class is the most natural proof. Though because of its length I am beginning to like the proof above. We now set into giving a complete write up of the proof of Theorem 1 outlined in class.

Let X be a metric space. We let $\mathcal{C}$ denote the collection of all Cauchy sequences in X. The following is the first crucial observation:

Lemma 3. *If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in X, then the sequence of real numbers $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to a non-negative real number.*

Proof. Since the real numbers are complete, it suffices to show that $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. To verify this, fix $\epsilon > 0$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy, there is an N so that $n, m > N$ implies

$$d(x_n, x_m) < \frac{\epsilon}{2} \quad \text{and} \quad d(y_n, y_m) < \frac{\epsilon}{2}.$$

We claim that for $n, m > N$, we have

$$|d(x_n, y_n) - d(x_m, y_m)| < \epsilon.$$

Observe that this equivalent to showing that for all $n, m > N$, we have

$$d(x_m, y_m) - \epsilon < d(x_n, y_n) < d(x_m, y_m) + \epsilon \tag{1}$$

To verify this, fix $n, m > N$. Observe that the triangle inequality implies that

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < \frac{\epsilon}{2} + d(x_m, y_m) + \frac{\epsilon}{2}.$$

This verifies the right inequality of equation 1. The other inequality works in a similar way. Observe again by the triangle inequality that

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m).$$

By algebra, this implies

$$d(x_n, y_n) \geq d(x_m, y_m) - d(x_m, x_n) - d(y_n, y_m) < d(x_m, y_m) - \frac{\epsilon}{2} - \frac{\epsilon}{2},$$

concluding the proof. □

We say two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ are asymptotic if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Before we move to far, note that there is a canonical way to produce asymptotic sequences:

Proposition 4 (Subsequences are asymptotic)**.** *Any subsequence of a Cauchy sequence $\{x_n\}$ is asymptotic to $\{x_n\}$.*

Proof. Let $\{x_n\}$ be Cauchy. Let $j : \mathbb{N} \to \mathbb{N}$ be an arbitrary strictly increasing function. Then, $\{x_{j(n)}\}$ is an arbitrary subsequence of $\{x_n\}$. We claim that

$$\lim_{n\to\infty} d(x_n, x_{j(n)}) = 0.$$

We verify this by definition. Let $\epsilon > 0$. Then since $\{x_n\}$ is Cauchy, there is an N so that $m, n > N$ implies $d(x_n, x_m) < \epsilon$. Since j is strictly increasing we know that $j(n) > n$. Thus, for $n > N$, we have $j(n) > n > N$ and so $d(x_n, x_{j(n)}) < \epsilon$. This verifies that $n > N$ implies $d(x_n, x_{j(n)}) < \epsilon$, so $\lim_{n\to\infty} d(x_n, x_{j(n)}) = 0$ by definition. □

A more important observation is the following:

Proposition 5. *The condition that two Cauchy sequences be asymptotic is an equivalence relation on $\mathcal{C}$.*

Proof of Lemma. First observe that the relation is reflexive. That is, every Cauchy sequence is asymptotic to itself, because

$$\lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} 0 = 0.$$

Now we verify symmetry. Suppose $\{x_n\}$ is asymptotic to $\{y_n\}$. Then,

$$0 = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n),$$

so $\{y_n\}$ is asymptotic to $\{x_n\}$.

Finally, we claim that it is transitive. This holds because of the triangle inequality. Suppose $\{x_n\}$ is asymptotic to $\{y_n\}$ and that $\{y_n\}$ is asymptotic to $\{z_n\}$. Then, by the triangle inequality,

$$0 \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n).$$

Observe that the right side tends to zero as $n \to \infty$, so by the squeeze theorem we have $\lim_{n\to\infty} d(x_n, z_n) = 0$. □

Because of the above proposition, we will say two asymptotic Cauchy sequences are *asymptotically equivalent*. We define $\tilde{X}$ to be the collection of asymptotic equivalence classes. We use $[\{x_n\}]$ to denote the equivalence class of a sequence $\{x_n\} \in \mathcal{C}$. Now consider the function

$$\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}; \quad \big([\{x_n\}], [\{y_n\}]\big) \mapsto \lim_{n\to\infty} d(x_n, y_n). \tag{2}$$

The limit on the right exists for any pair of Cauchy sequence, because of Lemma 3. However, it is not clear that the function is well defined, i.e., that the limit does not depend on the representatives taken form asymptotic equivalence classes. We will prove $\tilde{d}$ is well-defined using the following Proposition.

Proposition 6 (Alternation)**.** *Two Cauchy sequences* $\{x_n\}$ *and* $\{y_n\}$ *are asymptotic if and only if the alternating sequence* $\{z_n\}$ *defined by*

$$z_n = \begin{cases} x_{(n-1)/2} & \text{if } n \text{ is odd} \\ y_{n/2} & \text{if } n \text{ is even.} \end{cases}$$

is Cauchy.

Proof. First suppose $\{z_n\}$ is Cauchy. We claim $\{x_n\}$ and $\{y_n\}$ are asymptotic. To verify this, fix $\epsilon > 0$. We claim that there is an N so that $d(x_n, y_n) < \epsilon$ for $n > N$. Since $\{z_n\}$ is Cauchy, there is an N' so that $n, m > N'$ implies $d(z_n, z_m) < \epsilon$. Now let $N = N'/2$. Then if $n > N$ we have $2n + 1 > N'$ and $2n > N'$. Thus

$$d(z_{2n+1}, z_{2n}) = d(x_n, y_n) < \epsilon.$$

□

Proposition 7. *The function* $\tilde{d}$ *defined in equation 2 is well-defined.*

Proof. Suppose $\{x_n\}$ and $\{x'_n\}$ are asymptotic Cauchy sequences, and suppose also that $\{y_n\}$ and $\{y'_n\}$ are asymptotic. We must show that

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n).$$

Let $\{\tilde{z}_n\}$ denote the sequence which alternates between $\{x_n\}$ and $\{x'_n\}$ as in the Alternation Proposition. Also let $\{w_n\}$ be the sequence which alternates between $\{y_n\}$ and $\{y'_n\}$. By the alternation proposition, both $\{z_n\}$ and $\{w_n\}$ are Cauchy. Then by Lemma 3, there is an $L \in \mathbb{R}$ so that

$$\lim_{n\to\infty} d(z_n, w_n) = L.$$

Now observe that by definition of the alternating sequences, the sequences $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ and $\{d(x'_n, y'_n)\}_{n\in\mathbb{N}}$ are subsequences of $\{d(z_n, w_n)\}_{n\in\mathbb{N}}$. It follows that

$$\lim_{n\to\infty} d(x_n, y_n) = L \quad \text{and} \quad \lim_{n\to\infty} d(x'_n, y'_n) = L.$$

Since these limits are equal, the function $\tilde{d}$ is well-defined. □

Lemma 8. *The function* $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ *is a metric on* $\tilde{X}$*.*

Proof. We will check that $\tilde{d}$ satisfies the definition of a metric.

First, $\tilde{d}$ takes only non-negative values, because it is defined as a limit of non-negative numbers. We must show that $\tilde{d}(\tilde{x}, \tilde{y}) = 0$ if and only if $\tilde{x} = \tilde{y}$. This is essentially a tautology. Choose representative sequences $\{x_n\} \in \tilde{x}$ and $\{y_n\} \in \tilde{y}$. By definition of $\tilde{d}$ and asymptotic, we see $\tilde{d}(\tilde{x}, \tilde{y}) = 0$ if and only if $\{x_n\}$ and $\{y_n\}$ are asymptotic, which is equivalent to the statement that $\tilde{x} = \tilde{y}$ since these are asymptotic-equivalence classes.

To see $\tilde{d}$ is symmetric, let $\tilde{x}, \tilde{y} \in \tilde{X}$. Choose representatives $\{x_n\} \in \tilde{x}$ and $\{y_n\} \in \tilde{y}$. Observe

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \tilde{d}(\tilde{y}, \tilde{x}).$$

Last, we will check that $\tilde{d}$ satisfies the triangle inequality. Let $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$. Choose representatives $\{x_n\} \in \tilde{x}$, $\{y_n\} \in \tilde{y}$ and $\{z_n\} \in \tilde{z}$. Observe that by the triangle inequality on X, we have

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n).$$

Each of the terms here has a limit, so we conclude by properties of limits that

$$\lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n).$$

But, this just says that $\tilde{d}(\tilde{x}, \tilde{z}) \leq \tilde{d}(\tilde{x}, \tilde{y}) + \tilde{d}(\tilde{y}, \tilde{z})$. □

Now we need to verify that there is an isometry of X into $\tilde{X}$. If $x \in X$, let $\{c^x_n\}_{n\in\mathbb{N}}$ denote the constant sequence defined by $c^x_n = x$ for each $n \in \mathbb{N}$. We define

$$\phi : X \to \tilde{X}; \quad x \mapsto [\{c^x_n\}].$$

We claim the following aspect of the definition of completion.

Proposition 9. *The map ϕ is an isometry, and $\phi(X)$ is dense in $\tilde{X}$.*

Proof. First we observe that ϕ is an isometry. Let $x, y \in X$. Then,

$$\tilde{d}\big(\phi(x), \phi(y)\big) = \tilde{d}\big([\{c^x_n\}], [\{c^y_n\}]\big) = \lim_{n\to\infty} d(c^x_n, c^y_n) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

Now we will verify density. Choose $\tilde{x} \in X$, and choose a representative $\{x_n\} \in \tilde{x}$. We will show $\phi(X)$ is dense by producing an $y \in X$ so that $\tilde{d}\big(\tilde{x}, \phi(y)\big) < \epsilon$ for any $\epsilon > 0$. Fix $\epsilon > 0$. Then since $\{x_n\}$ is Cauchy, there is an N so that $n, m > N$ implies $d(x_n, x_m) < \frac{\epsilon}{2}$. Fix $y = x_m$ for some $n > N$. Recall $\phi(y) = [\{c^y_n = y\}_{n\in\mathbb{N}}]$. Then for $n > N$, we have

$$d(x_n, c^y_n) = d(x_n, y) = d(x_n, x_m) < \frac{\epsilon}{2}.$$

Therefore

$$\tilde{d}\big(\tilde{x}, \phi(y)\big) = \lim_{n\to\infty} d(x_n, c^y_n) \leq \frac{\epsilon}{2} < \epsilon$$

as required. □

The isometry ϕ has one more property that will turn out to be very useful to us.

Proposition 10. *Let $\{x_n\}$ be a Cauchy sequence and let $\tilde{x} = [\{x_n\}] \in \tilde{X}$ be its asyptotic-equivalence class. Then, $\{\phi(x_n)\}_{n\in\mathbb{N}}$ converges to $\tilde{x}$ in $\tilde{X}$.*

Proof. Let $\{x_n\}$ be a Cauchy sequence. Then for every $\epsilon > 0$, we can choose an $N_\epsilon \in \mathbb{R}$ so that $n, m > N_\epsilon$ implies $d(x_n, x_m) < \epsilon$.

By definition of $\tilde{x}$, the Cauchy sequence $\{x_n\}$ is a representative of the class. Observe that by definition of $\phi(x_n)$ and $\tilde{d}$, for any $k \in \mathbb{N}$,

$$\tilde{d}\big(\tilde{x}, \phi(x_k)\big) = \lim_{n\to\infty} d(x_n, x_k). \tag{3}$$

Note that this limit exists as a consequence of Lemma 3: both $\{x_n\}$ and a constant sequence are Cauchy. Observe that $k > N_\epsilon$ and $n > N_\epsilon$ implies that $d(x_n, x_k) < \epsilon$. Because equation 3 is a limit in n, for $k > N_\epsilon$, we have

$$\tilde{d}\big(\tilde{x}, \phi(x_k)\big) \leq \epsilon.$$

So, to see that $\{\phi(x_n)\}$ converges to $\tilde{x}$, observe that for any $\epsilon > 0$, whenever $k > N_{\epsilon/2}$ we have

$$\tilde{d}\big(\tilde{x}, \phi(x_k)\big) \leq \frac{\epsilon}{2} < \epsilon.$$

□

All that is left to prove Theorem 1 is to verify that $\tilde{X}$ is complete.

Lemma 11. *The metric space $(\tilde{X}, \tilde{d})$ is complete.*

As a first step in verifying this lemma, we make two observations about Cauchy sequences. First:

Proposition 12. *Let $\{y_n\}$ be a Cauchy sequence in a metric space (Y, d). Then, there is a strictly increasing function $j : \mathbb{N} \to \mathbb{N}$ so that the subsequence $\{y_{j(n)}\}$ satisfies*

$$d(y_{j(n)}, y_{j(m)}) < 2^{-\min(n,m)}.$$

Proof. Let $\{y_n\}$ be a Cauchy sequence. Let i be any natural number. Observe that because $\{y_n\}$ is Cauchy, there is an $N_i \in \mathbb{R}$ so that $n, m > N_i$ implies that $d(y_n, y_m) < 2^{-i}$. This defines a sequence $\{N_i\}_{i\in\mathbb{N}}$.

We use $\{N_i\}$ to define a strictly increasing sequence $\{j(i)\}_{i\in\mathbb{N}}$ inductively. As a base case, choose some $j(1) > N_1$. Now assume that $j(i)$ is defined. Choose $j(i+1)$ to be an integer satisfying

$$j(i+1) > \max\big(j(i), N_{i+1}\big).$$

Thus the sequence $\{j(i)\}_{i\in\mathbb{N}}$ is clearly strictly increasing and so defines a subsequence $\{y_{j(i)}\}_{i\in\mathbb{N}}$ of $\{y_n\}$.

Now we claim the inequality in the proposition holds. Let $m, n \in \mathbb{N}$. Then we can assume without loss of generality that $n \geq m$. Recall that $j(n) > N_n$ and since j is strictly increasing so that $j(m) > j(n) > N_n$. Then by definition of N_n, we have

$$d(y_{j(n)}, y_{j(m)}) < 2^{-n} = 2^{-\min(n,m)}.$$

□

The second observation concerns a property of convergent sequences of the type produced in Proposition 12.

Proposition 13. *Let $\{y_n\}$ be a sequence in a metric space (Y, d) converging to $y \in Y$ and suppose that for all $n, m \in \mathbb{N}$ we have*

$$d(y_n, y_m) < 2^{-\min(n,m)}.$$

Then, for each $n \in \mathbb{N}$, we have

$$d(y_n, y) \leq 2^{-n}.$$

Proof. Observe by continuity of the distance function,

$$d(y_n, y) = \lim_{m\to\infty} d(y_n, y_m).$$

Since $d(y_n, y_m) < 2^{-n}$ for $m > n$, the limit above must be non-strictly less than 2^{-n}. □

With those observations made, we can now prove that $(\tilde{X}, \tilde{d})$ is a complete metric space.

Proof of Lemma 11. Let $\{\tilde{x}^k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $\tilde{X}$. For each $k \in \mathbb{N}$ choose a representative $\{x^k_n\}_{n\in\mathbb{N}}$ from the class $\tilde{x}^k$. Recall that by Proposition refprop:subsequences, every subsequence of a Cauchy sequence is asymptotic to the sequence. So, in light of Proposition 12, we can assume (by possibly passing to a subsequence of $\{x^k_n\}$ for each k) that for all $k, n, m \in \mathbb{N}$, we have

$$d(x^k_n, x^k_m) < 2^{-\min(n,m)}. \tag{4}$$

For each $n \in \mathbb{N}$, define $y_n = x_n^n$. This defines a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X. We will show that $\{y_n\}$ is Cauchy and so defines an equivalence class $\tilde{y} = \{y_n\} \in \tilde{X}$. We will also show that $\{\tilde{x}^k\}$ converges to $\tilde{y}$, which will verify that $\tilde{X}$ is complete.

Recall that Proposition 9 showed that $\phi : X \to \hat{X}$ is an isometry defined by $\phi(x) = [\{c_n^x\}_{n\in\mathbb{N}}]$ where $c_n^x = x$ for all $n \in \mathbb{N}$. This is an isometry, so $\{y_n\}$ is a Cauchy sequence if and only if $\phi(y_n)$ is Cauchy. We will show that $\phi(y_n)$ is Cauchy.

Fix some $k \in \mathbb{N}$, and recall that we choose $\{x_n^k\}_{n\in\mathbb{N}}$ so that it satisfied the equation 4. Because ϕ is an isometry, we see that correspondingly,

$$\tilde{d}\big(\phi(x_n^k), \phi(x_m^k)\big) < 2^{-\min(n,m)}.$$

Recall that $\{x_n^k\} \in \tilde{x}^k$, and therefore $\phi(x_n^k)$ tends to $\tilde{x}^k$ as $n \to \infty$ by Proposition 10. Then by proposition 13, we see that

$$\tilde{d}\big(\phi(x_n^k), \tilde{x}^k\big) \leq 2^{-n}. \tag{5}$$

Note that this statement holds independent of k.

The proof that $\phi(y_n)$ is Cauchy is a standard $\epsilon/3$ proof. Fix some $\epsilon > 0$. Then we can choose an N so that $n > N$ implies $2^{-n} < \frac{\epsilon}{3}$. Also because $\{\tilde{x}^k\}$ is Cauchy, there is an K so that $k, \ell > K$ implies $\tilde{d}(\tilde{x}^k, \tilde{x}^\ell) < \frac{\epsilon}{3}$. Now choose $n, m > \max(N, K)$. By equation 5 and recalling that $y_n = x_n^n$, we have that

$$\tilde{d}\big(\phi(y_n), \tilde{x}^n\big) \leq 2^{-n} < \frac{\epsilon}{3} \quad \text{and} \quad \tilde{d}\big(\phi(y_m), \tilde{x}^m\big) \leq 2^{-m} < \frac{\epsilon}{3}.$$

Also, by our definition of K, we see that $\tilde{d}(\tilde{x}^m, \tilde{x}^n) < \frac{\epsilon}{3}$. Thus by the triangle inequality,

$$\tilde{d}\big(\phi(y_n), \phi(y_m)\big) \leq \tilde{d}\big(\phi(y_n), \tilde{x}^n\big) + \tilde{d}(\tilde{x}^n, \tilde{x}^m) + \tilde{d}\big(\tilde{x}^m, \phi(y_m)\big) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Now we know that $\{y_n\}$ is Cauchy. Let $\tilde{y} = [\{y_n\}]$. We claim that $\{\tilde{x}^n\}_{n\in\mathbb{N}}$ converges to $\tilde{y}$ as $n \to \infty$. First recall that $y_n = x_n^n$. Observe that by equation 5,

$$\tilde{d}\big(\tilde{x}^n, \phi(y_n)\big) = \tilde{d}\big(\tilde{x}^n, \phi(x_n^n)\big) \leq 2^{-n}.$$

Now to see that $\{\tilde{x}^n\}$ converges to $\tilde{y}$, fix an $\epsilon > 0$. We can choose an N_1 so that $n > N_1$ implies $2^{-n} < \frac{\epsilon}{2}$. Also recall that $\phi(y_n)$ converges to $\tilde{y}$ within $\tilde{X}$ by Proposition 10. Therefore, we can choose an N_2 so that $n > N_2$ implies

$$\tilde{d}\big(\phi(y_n), \tilde{y}\big) < \frac{\epsilon}{2}.$$

Observe that by the triangle inequality that if $n > \max(N_1, N_2)$, we have

$$\tilde{d}(\tilde{x}^k, \tilde{y}) \leq \tilde{d}\big(\tilde{x}^k, \phi(y_k)\big) + \tilde{d}(\phi(y_k), \tilde{y}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This verifies that $\{\tilde{x}^n\}$ converges to $\tilde{y}$ as required. □