THE INVERSE FUNCTION THEOREM

W. PATRICK HOOPER

The implicit function theorem is the following result:

Theorem 1. *Let f be a C^1 function from a neighborhood of a point $\mathbf{a} \in \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $A = Df(\mathbf{a})$ is invertible. Then, there is a neighborhood U of $\mathbf{a}$ so that the restriction $f|_U : U \to f(U)$ is invertible and $D(f|_U^{-1})\big(f(\mathbf{x})\big) = Df(\mathbf{x})^{-1}$ for all $x \in U$.*

We will pursue a proof of this fact using the contraction mapping theorem. (I'm not sure this is the quickest path to a proof, but we are following this approach to demonstrate the technique of this sort of proof.)

1. A family of contractions

The approach is motivated by Newton's method for finding roots of a polynomial (or other C^1 function) by iteratively improving approximations.

We begin with a discussion of the idea of the proof.

Our function f is C^1 at $\mathbf{a} \in \mathbb{R}^n$ with derivative A. Let $\mathbf{y}_*$ be a point nearby $\mathbf{b} = f(\mathbf{a})$, and let $\mathbf{x}_0$ be a point near $\mathbf{a}$. We treat $\mathbf{x}_0$ as a guess for the value of $f^{-1}(\mathbf{y}_*)$, and will try to improve this guess. Because f is C^1, Df is nearly A on a small neighborhood about $\mathbf{a}$. So an approximation for the function f near $\mathbf{x}_0$ (which is near $\mathbf{a}$) is given by the affine map

$$L(\mathbf{x}) = f(\mathbf{x}_0) + A(\mathbf{x} - \mathbf{x}_0).$$

Because A is invertible, we can invert the function L. We think of L^{-1} as an approximation to f^{-1} near $f(x_0)$, and we have

$$L^{-1}(\mathbf{y}) = \mathbf{x}_0 + A^{-1}\big(\mathbf{y} - f(\mathbf{x}_0)\big).$$

So our iteratively improved guess for the value of $f^{-1}(\mathbf{y}_*)$ is obtained by replacing $\mathbf{x}_0$ with $x_1 = L^{-1}(\mathbf{y}_*)$.

The notation above is not ideal, so we will introduce new notation. Let N be a neighborhood of $\mathbf{a}$ on which f is C^1. We will denote by $\phi_{\mathbf{y}_*}$ the map $\mathbf{x}_0 \mapsto L^{-1}(\mathbf{y}_*)$ with L^{-1} depending on $\mathbf{x}_0$ as above. Getting rid of the asterisk, we have:

$$\phi_{\mathbf{y}} : N \to \mathbb{R}^n; \quad \mathbf{x} \mapsto \mathbf{x} + A^{-1}\big(\mathbf{y} - f(\mathbf{x})\big).$$

We record some basic properties of this function in the following propositions. First

Proposition 2. *The map $\phi_{\mathbf{b}}$ fixes $\mathbf{a}$. Moreover, we have $f(\mathbf{x}) = \mathbf{y}$ if and only if $\mathbf{x}$ is fixed by $\phi_{\mathbf{y}}$.*

Proof. The first statement is a special case of the second. Observe that $\phi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x}$ if and only if

$$\mathbf{0} = \phi_{\mathbf{y}}(\mathbf{x}) - \mathbf{x} = A^{-1}\big(\mathbf{y} - f(\mathbf{x})\big).$$

Date: October 22, 2014.

By multiplying through by the invertible matrix A, we see that this is equivalent to $\mathbf{0} = \mathbf{y} - f(\mathbf{x})$. $\square$

The following explains that $\phi_{\mathbf{b}}$ is a very strong contraction locally near $\mathbf{a}$.

Proposition 3. *For any $c > 0$, there is an $r = r(c) > 0$ so that the closed ball $\bar{B}_r(\mathbf{a})$ of radius r centered at $\mathbf{a}$ is contained in the domain of f and whenever $\mathbf{x}, \mathbf{y} \in \bar{B}_r(\mathbf{a})$ we have*

$$|\phi_{\mathbf{b}}(\mathbf{x}) - \phi_{\mathbf{b}}(\mathbf{y})| \leq c|\mathbf{x} - \mathbf{y}|.$$

We will make use of the idea of the *operator norm* in the proof. If M is an $n \times n$ matrix, its *operator norm* is

$$\|M\| = \max_{\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n} \frac{|M\mathbf{x}|}{|\mathbf{x}|}.$$

The expression inside the maximum is invariant under scaling, so by continuity and compactness of the unit sphere, the supremum is realized at some point on this sphere. It easily follows that the operator norm has the following properties:

- The identity has operator norm 1.
- The quantity $\|M\|$ varies continuously in M.
- $\|M\| \geq 0$ with equality if and only M is the zero matrix.
- $\|cM\| = |c|\|M\|$ if M is $n \times n$ and $c \in \mathbb{R}$.
- $\|M_1 + M_2\| \leq \|M_1\| + \|M_2\|$ and $\|M_1 M_2\| \leq \|M_1\|\|M_2\|$ for every pair of $n \times n$ matrices.

Proof of Proposition 3. Choose an r_0 so that f is C^1 on $\bar{B}_{r_0}(\mathbf{a})$. Choose $\mathbf{x}, \mathbf{y} \in \bar{B}_{r_0}(\mathbf{a})$. Observe that

$$\phi_{\mathbf{b}}(\mathbf{x}) - \phi_{\mathbf{b}}(\mathbf{y}) = \mathbf{x} - \mathbf{y} - A^{-1}\big(f(\mathbf{x}) - f(\mathbf{y})\big) = -A^{-1}\big(f(\mathbf{x}) - f(\mathbf{y}) - A(\mathbf{x} - \mathbf{y})\big). \tag{1}$$

Consider the n-component functions of f, i.e.,

$$f(\mathbf{x}) = \big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})\big).$$

Let $\mathbf{z}_t = t\mathbf{x} + (1-t)\mathbf{y}$ for $x \in [0,1]$. The fundamental theorem of calculus in each coordinate tells us that

$$f_i(\mathbf{x}) - f_i(\mathbf{y}) = \int_0^1 \big[Df\big(\mathbf{z}_t\big)(\mathbf{x} - \mathbf{y})\big]_i \, dt.$$

By considering our integrals to be defined coordinate wise, we can write:

$$f(\mathbf{x}) - f(\mathbf{y}) = \int_0^1 Df\big(\mathbf{z}_t\big)(\mathbf{x} - \mathbf{y}) \, dt.$$

Plugging this into equation 1 and simplifying, we see:

$$\begin{aligned} \phi_{\mathbf{b}}(\mathbf{x}) - \phi_{\mathbf{b}}(\mathbf{y}) &= -A^{-1}\Big(\textstyle\int_0^1 Df\big(\mathbf{z}_t\big)(\mathbf{x} - \mathbf{y}) \, dt - A(\mathbf{x} - \mathbf{y})\Big) \\ &= -A^{-1}\Big(\textstyle\int_0^1 \big[Df\big(\mathbf{z}_t\big) - A\big](\mathbf{x} - \mathbf{y}) \, dt\Big). \end{aligned} \tag{2}$$

Now we will prove the proposition while making use of the operator norm. Fix some $c > 0$. Since $Df(\mathbf{a}) = A$, we have $\|Df(\mathbf{a}) - A\| = 0$. Because f is C^1 and by continuity of the operator norm, we can choose an r so that whenever $\mathbf{z} \in \bar{B}_r(\mathbf{a})$, we have

$$\|Df(\mathbf{z}) - A\| \leq \frac{c}{\|A^{-1}\|}.$$

Now take $\mathbf{x}, \mathbf{y} \in \bar{B}_r(\mathbf{a})$ and recall that with our definition above we have $\mathbf{z}_t \in \bar{B}_r(\mathbf{a})$ for every $t \in [0,1]$. Then by definition of the operator norm,

$$\left|\big[Df\big(\mathbf{z}_t\big) - A\big](\mathbf{x}-\mathbf{y})\right| \leq \frac{c}{\|A^{-1}\|}|\mathbf{x}-\mathbf{y}|.$$

We are integrating the vectors $\big[Df\big(\mathbf{z}_t\big)-A\big](\mathbf{x}-\mathbf{y})$ from 0 to 1, which you should think of as an average. Since every vector has a length uniformly bounded as above, the average has the same bound:

$$\left|\int_0^1 \big[Df\big(\mathbf{z}_t\big) - A\big](\mathbf{x}-\mathbf{y})\ dt\right| \leq \frac{c}{\|A^{-1}\|}|\mathbf{x}-\mathbf{y}|.$$

We obtain the right side of equation 2 by multiplying by A^{-1}, so by using the definition of the operator norm, we have:

$$\big|\phi_{\mathbf{b}}(\mathbf{x}) - \phi_{\mathbf{b}}(\mathbf{y})\big| \leq \|A^{-1}\| \left|\int_0^1 \big[Df\big(\mathbf{z}_t\big) - A\big](\mathbf{x}-\mathbf{y})\ dt\right| \leq c|\mathbf{x}-\mathbf{y}|.$$

□

We will now improve the prior proposition so that it holds for $\mathbf{y}$ near $\mathbf{b}$.

Proposition 4. *Suppose $0 < c < 1$, and define $r = r(c) > 0$ as in Proposition 3.*

(1) For any $\mathbf{y} \in \mathbb{R}^n$, whenever $\mathbf{x}_1, \mathbf{x}_2 \in \bar{B}_r(\mathbf{a})$, we have

$$|\phi_{\mathbf{y}}(\mathbf{x}_1) - \phi_{\mathbf{y}}(\mathbf{x}_2)| \leq c|\mathbf{x}_1 - \mathbf{x}_2|.$$

(2) There is an open ball $V = V(c)$ about $\mathbf{b}$ so that for any $\mathbf{y} \in V$, the image $\phi_{\mathbf{y}}\big(\bar{B}_r(\mathbf{a})\big)$ is contained in the open ball $B_r(\mathbf{a})$.

Proof. To see statement (1), choose $\mathbf{x}_1, \mathbf{x}_2 \in \bar{B}_r(\mathbf{a})$. Observe that

$$\phi_{\mathbf{y}}(\mathbf{x}_1) - \phi_{\mathbf{y}}(\mathbf{x}_2) = \phi_{\mathbf{b}}(\mathbf{x}_1) - \phi_{\mathbf{b}}(\mathbf{x}_2).$$

It follows using Proposition 3 that

$$|\phi_{\mathbf{y}}(\mathbf{x}_1) - \phi_{\mathbf{y}}(\mathbf{x}_2)| = |\phi_{\mathbf{b}}(\mathbf{x}_1) - \phi_{\mathbf{b}}(\mathbf{x}_2)| \leq c|\mathbf{x}_1 - \mathbf{x}_2|.$$

Now we consider statement (2). Observe that the function $(\mathbf{x}, \mathbf{y}) \mapsto \phi_{\mathbf{y}}(\mathbf{x})$ is jointly continuous. In particular, if $\bar{V}_0$ is a compact neighborhood of $\mathbf{y}$, the restricted function

$$\bar{B}_r(\mathbf{a}) \times \bar{V}_0 \to \mathbb{R}^n; (\mathbf{x}, \mathbf{y}) \mapsto \phi_{\mathbf{y}}(\mathbf{x})$$

is uniformly continuous. Therefore, there is a $\delta > 0$ so that for each $\mathbf{y} \in \bar{V}_0$ and each $\mathbf{x} \in \bar{B}_r(\mathbf{a})$,

$$|\mathbf{y} - \mathbf{b}| < \delta \quad \text{implies} \quad |\phi_{\mathbf{y}}(\mathbf{x}) - \phi_{\mathbf{b}}(\mathbf{x})| < (1-c)r.$$

Now recall that because $\phi_{\mathbf{b}}$ contracts $\bar{B}_r(\mathbf{a})$ by a factor of c and fixes $\mathbf{a}$, we have $\mathbf{x} \in \bar{B}_r(\mathbf{a})$ implies

$$|\phi_{\mathbf{b}}(\mathbf{x}) - \mathbf{a}| \leq c|\mathbf{x} - \mathbf{a}| \leq cr.$$

Then by the triangle inequality, if $\mathbf{y} \in \bar{V}_0$ and $|\mathbf{y} - \mathbf{b}| < \delta$, for any $\mathbf{x} \in \bar{B}_{r_0}(\mathbf{a})$, we have

$$|\phi_{\mathbf{y}}(\mathbf{x}) - \mathbf{a}| \leq |\phi_{\mathbf{y}}(\mathbf{x}) - \phi_{\mathbf{b}}(\mathbf{x})| + |\phi_{\mathbf{b}}(\mathbf{x}) - \mathbf{a}| < (1-c)r + cr = r.$$

So, the conclusion of statement (2) follows by choosing $V = V(c)$ to be an open ball about $\mathbf{b}$ that is contained in $\bar{V}_0 \cap B_\delta(\mathbf{b})$. □

For concreteness take $c_0 = \frac{1}{2}$ in the above proposition. Then there is an $r_0 = r(c_0)$ and a neighborhood $V_0 = V(c_0)$ so that for any $\mathbf{y} \in V_0$, $\phi_{\mathbf{y}}$ sends $\bar{B}_{r_0}(\mathbf{a})$ into its interior and contracts distances by a factor of at least $\frac{1}{2}$. As a consequence of the contraction mapping theorem, we see that for each $\mathbf{y} \in V$ (with V as in Proposition 4), there is a unique fixed point of the restriction of $\phi_{\mathbf{y}}$ to $\bar{B}_{r_0}(\mathbf{a})$, and this fixed point lies in the open ball $B_{r_0}(\mathbf{a})$. Let

$$g : V_0 \to B_{r_0}(\mathbf{a})$$

be the map which sends each $\mathbf{y} \in V_0$ to this fixed point of $\phi_{\mathbf{y}}$. As a direct consequence of Proposition 2:

Corollary 5. *We have $f \circ g(\mathbf{y}) = \mathbf{y}$ for every $\mathbf{y} \in V_0$.*

We now turn our attention to proving that g is continuous. The key point is that the proof of the contraction mapping theorem gives information about how close a point $\mathbf{x}$ is to the fixed point of a contraction. We make use of the following general result.

Lemma 6. *Let X be a metric space and $\phi : X \to X$ be a contraction by c with $0 < c < 1$. Let $y \in X$ be a (necessarily unique) fixed point of ϕ and let $x_0 \in X$ be arbitrary. Then*

$$d(y, x_0) \leq \frac{1}{1-c} d(x_0, x_1).$$

Proof. Extend the definition of x_0 inductively by defining $x_{i+1} = \phi(x_i)$ for each integer $i \geq 0$. Recall that $\{x_i\}$ tends to the fixed point y. Then, by the triangle inequality,

$$d(x_0, y) = \lim_{j \to \infty} d(x_0, x_j) \leq \lim_{j \to \infty} \sum_{i=0}^{j-1} d(x_i, x_{i+1}) = \sum_{i=0}^{\infty} d(x_i, x_{i+1}).$$

Now observe that $d(x_i, x_{i+1}) \leq c^i d(x_0, x_1)$. We conclude that

$$d(x_0, y) \leq d(x_0, x_1) \sum_{i=0}^{\infty} c^i = \frac{1}{1-c} d(x_0, x_1).$$

□

Proposition 7. *The function $g : V_0 \to B_{r_0}(\mathbf{a})$ which sends $\mathbf{y}$ to the fixed point of $\phi_{\mathbf{y}}$ is continuous. In fact,*

$$|g(\mathbf{y}_1) - g(\mathbf{y}_2)| \leq 2\|A^{-1}\||\mathbf{y}_1 - \mathbf{y}_2|, \quad \textit{for each } \mathbf{y}_1, \mathbf{y}_2 \in V_0.$$

Proof. The second statement directly implies the continuity of g. Choose $\mathbf{y}_1, \mathbf{y}_2 \in V_0$ so that Set $\mathbf{x}_1 = g(\mathbf{y}_1)$ and $\mathbf{x}_2 = g(\mathbf{y}_2)$. We claim $|\mathbf{x}_1 - \mathbf{x}_2| < \epsilon$. Observe that

$$\phi_{\mathbf{y}_1}(\mathbf{x}_2) - \mathbf{x}_2 = A^{-1}\big(\mathbf{y}_1 - f(\mathbf{x}_2)\big) = A^{-1}(\mathbf{y}_1 - \mathbf{y}_2).$$

Thus, $|\phi_{\mathbf{y}_1}(\mathbf{x}_2) - \mathbf{x}_2| \leq \|A^{-1}\||\mathbf{y}_1 - \mathbf{y}_2|$. Recall $\mathbf{x}_1$ is the fixed point of $\phi_{\mathbf{y}_1}$, which contracts distances by $\frac{1}{2}$. Thus by the lemma above,

$$|\mathbf{x}_1 - \mathbf{x}_2| \leq \frac{1}{1 - \frac{1}{2}} \|A^{-1}\||\mathbf{y}_1 - \mathbf{y}_2| = 2\|A^{-1}\||\mathbf{y}_1 - \mathbf{y}_2|.$$

□

Now we address differentiability. Note that it suffices to prove that $Dg(\mathbf{b}) = Df(\mathbf{a})^{-1}$. This is because to check this at a point $f(\mathbf{x}) \neq \mathbf{b}$, we can repeat the argument above to obtain a local inverse defined in a neighborhood of $f(\mathbf{x})$. Since the inverses must agree on their intersection, the same argument implies that $Dg\big(f(\mathbf{x})\big) = Df(\mathbf{x})^{-1}$. From this observation, the following Lemma completes the proof:

Lemma 8. *The function g is differentiable at $\mathbf{b}$ and $Dg(\mathbf{b}) = A^{-1}$.*

Proof. We will verify the definition of differentiable. We must show that

$$\lim_{\mathbf{y}\to\mathbf{b}} \frac{\big|g(\mathbf{y}) - [\mathbf{a} + A^{-1}(\mathbf{y}-\mathbf{b})]\big|}{|\mathbf{y}-\mathbf{b}|} = 0.$$

To verify this, it suffices to show that for any $\epsilon > 0$ there is a c with $0 < c < 1$ so that for any point $\mathbf{y}$ in the neighborhood $V = V(c)$ defined in Proposition 4, we have

$$\frac{\big|g(\mathbf{y}) - [\mathbf{a} + A^{-1}(\mathbf{y}-\mathbf{b})]\big|}{|\mathbf{y}-\mathbf{b}|} < \epsilon. \tag{3}$$

Given $\epsilon > 0$, we choose c so that $2c\|A^{-1}\| < \epsilon$ and $0 < c \leq \frac{1}{2}$. Let $r = r(c)$ and $V = V(c)$. Choose $\mathbf{y} \in V_c$ and let $\mathbf{x} = g(\mathbf{y}) \in B_r(\mathbf{a})$. As in the prior proof, we observe

$$\phi_{\mathbf{b}}(\mathbf{x}) - \mathbf{x} = A^{-1}\big(\mathbf{b} - f(\mathbf{x})\big) = A^{-1}(\mathbf{b}-\mathbf{y}).$$

Observe that this relates to the numerator of the expression in equation 3:

$$g(\mathbf{y}) - [\mathbf{a} + A^{-1}(\mathbf{y}-\mathbf{b})] = \mathbf{x} + A^{-1}(\mathbf{b}-\mathbf{y}) - \mathbf{a} = \phi_{\mathbf{b}}(\mathbf{x}) - \mathbf{a}.$$

Now recall that $\phi_{\mathbf{b}}$ contracts distances in $B_r(\mathbf{a})$ by a factor of c and fixes $\mathbf{a}$, so

$$\big|g(\mathbf{y}) - [\mathbf{a} + A^{-1}(\mathbf{y}-\mathbf{b})]\big| = |\phi_{\mathbf{b}}(\mathbf{x}) - \mathbf{a}| \leq c|\mathbf{x}-\mathbf{a}|.$$

In summary, for $\mathbf{y} \in V$, we have $\mathbf{x} = g(\mathbf{y}) \in B_r(\mathbf{a})$ and

$$\frac{\big|g(\mathbf{y}) - [\mathbf{a} + A^{-1}(\mathbf{y}-\mathbf{b})]\big|}{|\mathbf{y}-\mathbf{a}|} \leq \frac{c|\mathbf{x}-\mathbf{a}|}{|\mathbf{y}-\mathbf{b}|} = \frac{c|g(\mathbf{y}) - g(\mathbf{b})|}{|\mathbf{y}-\mathbf{b}|}.$$

Then by Proposition 7, we have $|g(\mathbf{y}) - g(\mathbf{b})| \leq 2c\|A^{-1}\|$. Thus,

$$\frac{\big|g(\mathbf{y}) - [\mathbf{a} + A^{-1}(\mathbf{y}-\mathbf{b})]\big|}{|\mathbf{y}-\mathbf{a}|} \leq \frac{2c\|A^{-1}\||\mathbf{y}-\mathbf{b}|}{|\mathbf{y}-\mathbf{b}|} = 2c\|A^{-1}\| < \epsilon.$$

□