Math 70100: Functions of a Real Variable I
Homework 9, due Wednesday, November 12th.

1. *(Folland §1.5 # 30)* Let λ be Lebesgue measure on $\mathbb{R}$, and let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$. Show that for any $\alpha < 1$, there is an open interval I so that $\lambda(E \cap I) > \alpha\lambda(I)$. (*Hint:* An open set in $\mathbb{R}$ is a countable union of disjoint open intervals.)

Solution: Suppose the statement were false. Then there is a Lebesgue measureable set E with $\lambda(E) > 0$ and an α with $0 < \alpha < 1$ so that

$$\lambda(E \cap I) \leq \alpha\lambda(I) \quad \text{for any open interval } I \subset \mathbb{R}. \tag{1}$$

Since $\frac{1}{\alpha} > 1$, we can find a open set A containing E so that

$$\lambda(A) < \frac{1}{\alpha}\lambda(E). \tag{2}$$

(Actually this is true for any E if we use Lebesgue outer measure above.) Recall that any open set is a countable union of disjoint open intervals. Let $\{A_i\}$ be those intervals. Because of equation 1, we observe that

$$\lambda(E) = \sum_i \lambda(E \cap A_i) \leq \sum_i \alpha\lambda(A_i) \leq \alpha\lambda(A).$$

But, this is a contradiction since by including equation 2, we see

$$\lambda(E) \leq \alpha\lambda(A) < \lambda(E).$$

2. Imitate the construction of the middle-thirds Cantor set to show that for every c with $0 < c < 1$, there is a Cantor set $K \subset [0, 1]$ whose Lebesgue measure is c.

Solution: Choose a c with $0 < c < 1$. Let $\{b_n \in \mathbb{R} \ : \ n \in \mathbb{N}\}$ be any strictly decreasing sequence of real numbers with $b_1 < 1$ and $\lim_{n\to\infty} b_n = c$.

In order to define our Cantor set, we define the following maps. If $[a, b]$ is a closed interval, and $\ell < b - a$ is a real number, we define

$$L([a, b], \ell) = [a, a + \ell] \quad \text{and} \quad R([a, b], \ell) = [b - \ell, b].$$

Observe that so long as $2\ell < b - a$, the intervals $L([a, b], \ell)$ and $R([a, b], \ell)$ are equal sized intervals obtained by removing the middle open interval of the length $b - a - 2\ell$ from $[a, b]$.

We will now inductively define some collections of closed intervals. Define $\mathcal{I}_0 = \{[0, 1]\}$. Now suppose $\mathcal{I}_i$ is defined, we define

$$\mathcal{I}_{i+1} = \left\{L(I, \frac{b_{i+1}}{2^{i+1}}) \ : \ I \in \mathcal{I}_i\right\} \cup \left\{R(I, \frac{b_{i+1}}{2^{i+1}}) \ : \ I \in \mathcal{I}_i\right\}.$$

Observe that each $\mathcal{I}_i$ consists of 2^i intervals each of length $\frac{b_i}{2^i}$. Because $b_{i+1} < b_i$, the intervals of $\mathcal{I}_{i+1}$ are obtained by removing some middle open interval from the intervals of $\mathcal{I}_i$.

For each i, define $C_i = \bigcup\{I \in \mathcal{I}_i\}$. Then C_i is a disjoint union of 2^i intervas of length $\frac{b_i}{2^i}$. Therefore, $\lambda(C_i) = b_i$. The set $C = \bigcap_i C_i$ is a Cantor set, and by continuity of measure,

$$\lambda(C) = \lim_{i\to\infty} \lambda(C_i) = \lim_{i\to\infty} b_i = c.$$

3. *(Borel-Cantelli Lemma)* Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$. Suppose the sequence of sets $\{E_n \in \Sigma \;:\; n \in \mathbb{N}\}$ satisfies $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Show that the set

$$A = \{x \in X \;:\; \text{there are infinitely many } n \in \mathbb{N} \text{ so that } x \in E_n\}$$

is measurable (lies in Σ) and has measure zero.

Solution: For $N \in \mathbb{N}$, define $B_N = \bigcup_{n=N}^{\infty} E_n$. Then $B_N \in \Sigma$ and

$$A = \bigcap_{N=1}^{\infty} B_N \in \Sigma.$$

Observe that by countable subadditivity, we have

$$\mu(B_N) \le \sum_{n=N}^{\infty} \mu(E_n).$$

Since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, we have $\mu(B_N) \to 0$ as $N \to \infty$. Then by continuity of measure,

$$\mu(A) = \lim_{N\to\infty} \mu(B_N) = 0.$$

4. Construct a Lebesgue measurable set $E \subset [0, 1]$ of Lebesgue measure $\frac{1}{2}$ such that both E and $[0, 1] \setminus E$ are dense in $[0, 1]$.

5. *(Folland §1.2 # 10)* Show that if (X, Σ, μ) is a measure space and $E \in \Sigma$, then the function $\mu_E : \Sigma \to [0, \infty]$ defined by $\mu_E(A) = \mu(A \cap E)$ is a measure.

Solution: We need to check that $\mu_E(\emptyset) = 0$ and that μ_E is countably additive.

First observe that

$$\mu_E(\emptyset) = \mu(\emptyset \cap E) = \mu(\emptyset) = 0,$$

since μ is a measure.

Now suppose that $\{A_i \in \Sigma\}$ is a disjoint and countable. Let $A = \bigcup_i A_i$. Observe that $A \cap E = \bigcup_i (A_i \cap E)$. The collection $\{A_i \cap E\}$ is countable and disjoint, so because μ is a measure, we have

$$\mu_E(A) = \mu(A \cap E) = \sum_i \mu(A_i \cap E) = \sum_i \mu_E(A_i).$$

So, μ_E is countably additive.

6. *(Based on Folland §1.2 # 7)* Let Σ be a σ-algebra on the set X. Show that the collection of all measures on (X, Σ) is a closed convex cone in the sense that if μ_1 and μ_2 are measures on (X, Σ) and $c_1, c_2 \geq 0$, then so is $c_1\mu_1 + c_2\mu_2$. Are these measures closed under countable sums?

Solution: Let μ_1 and μ_2 be measures on (X, Σ) and let $c_1, c_2 \geq 0$. We will show that $\nu = c_1\mu_1 + c_2\mu_2$ is a measure. Observe that

$$\nu(\emptyset) = c_1\mu_1(\emptyset) + c_2\mu_2(\emptyset) = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

Also, let $\{A_i\} \subset \Sigma$ be a countable disjoint collection whose union is A, then by countable additivity applied to μ_1 and μ_2 has

$$\begin{aligned} \nu(A) &= c_1\mu_1(A) + c_2\mu_2(A) = c_1 \sum_i \mu_1(A_i) + c_2 \sum_i \mu_2(A_i) \\ &= \sum_i \big(c_1\mu_1(A_i) + c_2\mu_2(A_i)\big) = \sum_i \nu(A_i). \end{aligned}$$

This shows ν is a measure.

Now suppose that $\{\mu_i\}$ is a countable collection of measures on (X, Σ). Let $\nu = \sum_i \mu_i$. We have

$$\nu(\emptyset) = \sum_i \mu_i(\emptyset) = \sum_i 0 = 0.$$

Now suppose that $\{A_j\} \subset \Sigma$ is a countable collection of measurable sets, and let A be the union of these sets. Then by countable additivity of each μ_i,

$$\nu(A) = \sum_i \mu_i(A) = \sum_i \sum_j \mu_i(A_j).$$

We will show below that we can switch the order of this sum. So, we have

$$\nu(A) = \sum_j \sum_i \mu_i(A_j) = \sum_j \nu(A_j).$$

This proves that ν is a measure.

In order to switch the sum above, we will show that whenever $\{a_{i,j} \,:\, i,j \in \mathbb{N}\}$ is a collection of non-negative real numbers, we have

$$\sum_i \sum_j a_{i,j} = \sum_j \sum_i a_{i,j}.$$

To do this, by symmetry it suffices to show that

$$\sum_i \sum_j a_{i,j} \geq \sum_j \sum_i a_{i,j}.$$

To check this, let M be any number strictly less than the right hand side, i.e.,

$$M < \sum_j \sum_i a_{i,j}. \tag{3}$$

Since M is arbitrary, it suffices to prove that

$$\sum_i \sum_j a_{i,j} \geq M. \tag{4}$$

By equation 3, we see that there is a J so that

$$M < \sum_{j=1}^{J} \sum_{i=1}^{\infty} a_{i,j}.$$

Let $b_j = \sum_{i=1}^{\infty} a_{i,j}$. Let $\epsilon = (\sum_{j=1}^{J} b_j) - M > 0$. For each $j \in \{1, \ldots, J\}$ there is an $I_j \in \mathbb{N}$ so that

$$\sum_{i=1}^{I_j} a_{i,j} > b_j - \frac{\epsilon}{J}.$$

Now let $I = \max\{I_1, \ldots, I_J\} \in \mathbb{N}$. Then,

$$\sum_{j=1}^{J} \sum_{i=1}^{I} a_{i,j} > \sum_{j=1}^{J} (b_j - \frac{\epsilon}{J}) = M.$$

Now because we are only working with finite sums, we have

$$M < \sum_{i=1}^{I} \sum_{j=1}^{J} a_{i,j} \leq \sum_{i=1}^{I} \sum_{j=1}^{\infty} a_{i,j} \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}.$$

This verifies equation 4 as required.

7. *(Folland §1.4 # 18)* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra on X. Let $\mathcal{A}_\sigma \subset \mathcal{P}(X)$ denote the collection of all countable unions of sets in $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ denote the collection of all countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and let μ^* be the induced outer measure.

   (a) Show that for any $E \subset X$ and any $\epsilon > 0$, there is an $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq$

$\mu^*(E) + \epsilon$.

Solution: If $\mu^*(E) = \infty$, then we can take $A = X$, which lies in $\mathcal{A}$ because it is an algebra. By monotonicity, we also have $\mu^*(X) = \infty$.

Now assume $\mu^*(E) < \infty$ and let $\epsilon > 0$. Then by definition, $\mu^*(E)$ is the infimum over all countable unions $A = \bigcup_i A_i \in A_\sigma$ of $\sum_i \mu_0(A_i)$. So, in particular, we can find such an A so that $\sum_i \mu_0(A_i) < \mu^*(E) + \epsilon$. Recall $\mu^*(A_i) = \mu_0(A_i)$ since each $A_i \in \mathcal{A}$. By applying monotonicity and countable subadditivity, we see

$$\mu^*(E) \leq \mu^*(A) \leq \sum_i \mu_0(A_i) < \mu(E) + \epsilon.$$

(b) Suppose $\mu^*(E) < \infty$. Show that E is measurable if and only if there is a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

Solution: First suppose $E \subset X$ is measurable and $\mu^*(E) < \infty$. From the previous part, for any $n \in \mathbb{N}$, there is an $A_n \in A_\sigma$ containing E so that $\mu^*(A_n) < \mu^*(E) + \frac{1}{n}$. Define $B = \cap_n A_n \in \mathcal{A}_{\sigma\delta}$. Then by monotonicity twice, we see for each n that

$$\mu^*(E) \leq \mu^*(B) \leq \mu^*(A_n) < \mu^*(E) + \frac{1}{n}.$$

Since this holds for arbitrary n, we see $\mu^*(B) = \mu^*(E)$. Now because E is measurable, we have

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \setminus E).$$

Observe that $B \cap E = E$ and so $\mu^*(B) = \mu^*(B \cap E)$. Therefore $\mu^*(B \setminus E)$.

Let $E \subset X$. This time contrary to the statement of the problem, we will not assume that $\mu^*(E) < \infty$. Suppose there is a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$. We will verify that E is measurable by definition. Let $A \subset X$ be arbitrary. We need to check that

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E).$$

By subadditivity, we get clearly have

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \setminus E).$$

So, it remains to show the other inequality ($\geq$). Observe that B is measurable since it lies in the σ-algebra generated by $\mathcal{A}$. Therefore we know that

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B). \tag{5}$$

Observe that $A \cap E \subset A \cap B$, so we see

$$\mu^*(A \cap B) \geq \mu^*(A \cap E). \tag{6}$$

On the other hand, by measurability of B again, we have

$$\mu^*(A \setminus E) = \mu^*\big((A \setminus E) \setminus B\big) + \mu^*\big((A \setminus E) \cap B\big). \tag{7}$$

Observe that $(A \setminus E) \cap B = A \cap (B \setminus E)$ and $\mu^*(B \setminus E) = 0$ by assumption. Therefore, by monotonicity, we see that $\mu^*\big((A \setminus E) \cap B\big) = 0$. Also we can simplify $(A \setminus E) \setminus B$ as $A \setminus B$, since $E \subset B$. So equation 7 can be rewritten as

$$\mu^*(A \setminus E) = \mu^*\big(A \setminus B) \tag{8}$$

By combining equations 5, 6 and 8, we see

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B) \geq \mu^*(A \cap E) + \mu^*(A \setminus E).$$

We conclude that E is measurable.

(c) Recall that μ_0 is *σ-finite* if there is a countable collection $\{C_i\} \subset \mathcal{A}$ with $\bigcup_i C_i = X$ and $\mu_0(C_i) < \infty$ for all i. Show that if μ_0 is σ-finite, then even if $\mu^*(E) = \infty$ the statement from part (b) still holds.

Solution: We assume μ_0 is σ-finite and $\{C_i\}$ are as given in the problem. If the collection is finite, then X would have finite outer measure, so we can assume that $\{C_i\}$ is indexed by the natural numbers.

We will briefly describe the standard trick which allows us to make the collection $\{C_i\}$ disjoint. For each $i \in \mathbb{N}$ define

$$D_i = C_i \setminus \bigcup_{j<i} C_j.$$

Observe that these sets are pairwise disjoint, cover X, and $D_i \subset C_i$ so by monotonicity $\mu^0(C_i) < \mu^0(D_i)$.

Suppose E is measurable and $\mu^*(E) = \infty$. Since each $E \cap D_i$ is measurable, by part (a), for each i and each n there is an $A_{n,i} \in \mathcal{A}_\sigma$ containing $E \cap D_i$ so that

$$\mu^*(A_{n,i}) < \mu^*(E \cap D_i) + \frac{1}{n2^i}.$$

Observe that because $E \cap D_i$ is measurable and they are pairwise disjoint, we have $\sum_i \mu^*(E \cap D_i) = \mu^*(E)$. Set $A_n = \bigcup_i A_{n,i}$, which also lies in $\mathcal{A}_\sigma$. Now observe that

$$A_n \setminus E = \bigcup_i (A_{n,i} \setminus E) \subset \bigcup_i \big(A_{n,i} \setminus (E \cap D_i)\big).$$

So by countable subadditivity and measurability,

$$\mu^*(A_n \setminus E) \leq \sum_i \mu^*\big(A_{n,i} \setminus (E \cap D_i)\big) = \sum_i \mu^*(A_{n,i}) - \mu^*(E \cap D_i) < \sum_i \frac{1}{n2^i} = \frac{1}{n}.$$

Now set $B = \bigcap_n A_n \in \mathcal{A}_{\sigma\delta}$, which contains E. By monotonicity, we have

$$\mu^*(B \setminus E) \leq \mu^*(A_n \setminus E) < \frac{1}{n},$$

for each n, so $\mu^*(B \setminus E) = 0$.

For the converse, observe that the proof we gave of the converse in part (b) did not assume that $\mu^*(E) < \infty$. Indeed σ-finiteness of μ_0 is not needed for the converse.