Math 70100: Functions of a Real Variable I
Homework 9, due Wednesday, November 12th.

1. *(Folland §1.5 # 30)* Let λ be Lebesgue measure on $\mathbb{R}$, and let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$. Show that for any $\alpha < 1$, there is an open interval I so that $\lambda(E \cap I) > \alpha\lambda(I)$. (*Hint:* An open set in $\mathbb{R}$ is a countable union of disjoint open intervals.)
2. Imitate the construction of the middle-thirds Cantor set to show that for every c with $0 < c < 1$, there is a Cantor set $K \subset [0,1]$ whose Lebesgue measure is c.
3. *(Borel-Cantelli Lemma)* Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$. Suppose the sequence of sets $\{E_n \in \Sigma \ : \ n \in \mathbb{N}\}$ satisfies $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Show that the set

   $$A = \{x \in X \ : \ \text{there are infinitely many } n \in \mathbb{N} \text{ so that } x \in E_n\}$$

   is measurable (lies in Σ) and has measure zero.
4. Construct a Lebesgue measurable set $E \subset [0,1]$ of Lebesgue measure $\frac{1}{2}$ such that both E and $[0,1] \setminus E$ are dense in $[0,1]$.
5. *(Folland §1.2 # 10)* Show that if (X, Σ, μ) is a measure space and $E \in \Sigma$, then the function $\mu_E : \Sigma \to [0, \infty]$ defined by $\mu_E(A) = \mu(A \cap E)$ is a measure.
6. *(Based on Folland §1.2 # 7)* Let Σ be a σ-algebra on the set X. Show that the collection of all measures on (X, Σ) is a closed convex cone in the sense that if μ_1 and μ_2 are measures on (X, Σ) and $c_1, c_2 \geq 0$, then so is $c_1\mu_1 + c_2\mu_2$. Are these measures closed under countable sums?
7. *(Folland §1.4 # 18)* Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra on X. Let $\mathcal{A}_\sigma \subset \mathcal{P}(X)$ denote the collection of all countable unions of sets in $\mathcal{A}$, and let $\mathcal{A}_{\sigma\delta}$ denote the collection of all countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and let μ^* be the induced outer measure.
   (a) Show that for any $E \subset X$ and any $\epsilon > 0$, there is an $A \in \mathcal{A}_\sigma$ with with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.
   (b) Suppose $\mu^*(E) < \infty$. Show that E is measurable if and only if there is a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
   (c) Recall that μ_0 is *σ-finite* if there is a countable collection $\{C_i\} \subset \mathcal{A}$ with $\bigcup_i C_i = X$ and $\mu_0(C_i) < \infty$ for all i. Show that if μ_0 is σ-finite, then even if $\mu^*(E) = \infty$ the statement from part (b) still holds.