Math 70100: Functions of a Real Variable I
Homework 8, due Wednesday, November 5.

Name: Insert your name here.

1. *(Combines Pugh's Ch. 6 # 1-2)* Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. Prove that $m^* \circ f(A) = |a| \cdot m^*(A)$ for each $A \subset \mathbb{R}$, where m^* is the Lebesgue outer measure on $\mathbb{R}$.

Solution: Let $B = f(A)$. First we deal with the case when $a = 0$. Then $B = \emptyset$ if $A = \emptyset$ and $B = \{b\}$ otherwise. In either case, we see $B \subset (b - \epsilon, b + \epsilon)$, so

$$m^*(B) \leq \inf_{\epsilon > 0} |(b - \epsilon, b + \epsilon)| = \inf_{\epsilon > 0} 2\epsilon = 0.$$

Since $m^*(B) \geq 0$ by definition, we see $m^*(B) = 0$. This agrees with the formula provided since $a = 0$.

Now suppose $a \neq 0$. Observe $\{I_k\}$ is a countable cover of A by open intervals if and only if $\{f(I_k)\}$ is a countable cover of $B = f(A)$ by open intervals. Furthermore,

$$|f(I_k)| = |a||I_k|.$$

Therefore,

$$m^*(B) = \inf\{\sum_k |f(I_k)|\} = \inf\{\sum_k |a| \cdot |I_k|\} = |a| \inf\{\sum_k |I_k|\} = m^*(A).$$

where the infima are taken over all countable covers $\{I_k\}$ of A by open intervals.

2. Use the formula from the prior problem to show that the middle third Cantor set C satisfies $m^*(C) = 0$, where m^* is Lebesgue outer measure. (*Hint:* Use the self-similarity.)

Solution: Observe that $C \subset [0, 1]$ so that $m^*(C) \leq 1$. Observe that the two maps

$$f_1(x) = \frac{x}{3} \quad \text{and} \quad f_2(x) = \frac{x + 2}{3}$$

restrict to bijections from C to the left and right half of C, respectively. Therefore $m^* \circ f_i(C) = \frac{1}{3} m^*(C)$ for each $i \in \{1, 2\}$. By subadditivity,

$$m^*(C) \leq m^* \circ f_1(C) + m^* \circ f_2(C) = \frac{2}{3} m^*(C).$$

By solving for $m^*(C)$, we see $m^*(C) \leq 0$. Therefore $m^*(C) = 0$.

3. *(Royden §2.2 # 7)* A set of real numbers is said to be a G_δ set if it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which $E \subset G$ and $m^*(G) = m^*(E)$.

Solution: Let E be a bounded set. Then because E can be contained in a bounded interval, $m^*(E) < \infty$. By definition of m^*, for each integer $n \geq 1$, there is a countable covering $\mathcal{I}^n = \{I_k^n\}$ by open intervals so that

$$\sum_k |I_k^n| \leq m^*(E) + \frac{1}{n}.$$

For each n, consider the open set $U_n = \bigcup_k I_k^n$, which by construction contains E. Define $G = \bigcap_n U_n$, which is a G_δ set. Then $E \subset G$. By monotonicity of m^*, we have $m^*(E) \leq m^*(G)$. Furthermore, $\mathcal{I}^n$ is a covering of G for all n. Therefore

$$m^*(G) \leq \inf_n \sum_k |I_k^n| \leq \inf_n \big(m^*(E) + \frac{1}{n}\big) = m^*(E).$$

It follows that we have $m^*(E) = m^*(G)$.

4. Fix some real number $d \geq 0$. For a subset $A \subset \mathbb{R}$ and $\delta > 0$, let

$$H_\delta^d(A) = \inf \big\{ \sum_k |I_k|^d \big\},$$

where the infimum is taken over all countable covers $\{I_k\}$ of A by open intervals each of which has length less than δ. The *d-dimensional Hausdorff outer measure* of A is

$$H^d(A) = \lim_{\delta \to 0} H_\delta^d(A).$$

You can use without proof that H^d is an outer measure. You may also use without proof that when $d = 1$, H^d is the Lebesgue outer measure on $\mathbb{R}$.

(a) Explain why if $\delta < \delta'$, then $H_\delta^d(A) \geq H_{\delta'}^d(A)$ for every $A \subset \mathbb{R}$. (*Remark*: It follows that $H^d(A) = \sup_{\delta > 0} H_\delta^d(A)$.)

Solution: Fix $A \subset \mathbb{R}$ and $d \geq 0$. For each $\delta > 0$, let $\mathcal{C}_\delta$ denote the collection of all covers of A by open intervals each of which has diameter less than δ. Observe that if $\delta < \delta'$, then $\mathcal{C}_\delta \subset \mathcal{C}_{\delta'}$. Therefore,

$$H_\delta^d(A) = \inf_{\mathcal{I} \in \mathcal{C}_\delta} \big\{ \sum_{I \in \mathcal{I}} |I|^d \big\} \geq \inf_{\mathcal{I} \in \mathcal{C}_{\delta'}} \big\{ \sum_{I \in \mathcal{I}} |I|^d \big\} = H_{\delta'}^d(A),$$

since the first infimum is taken over a smaller set of values.

(b) Show that $H^d([0,1]) = 0$ for every $d > 1$.

Solution: Let $d > 1$. Let $\delta > 0$. We will show that $H_\delta^d([0,1]) = 0$. For each integer $n \geq 0$ and each $\epsilon > 0$, define the covering

$$\mathcal{I}_{n,\epsilon} = \{(\frac{i}{n} - \epsilon, \frac{i+1}{n} + \epsilon) \ : \ 0 \leq i < n\},$$

which is a covering of $[0,1]$ by n open intervals each with length $\frac{1}{n} + 2\epsilon$. Then,

$$H^d_\delta([0,1]) \leq n(\frac{1}{n} + 2\epsilon)^d$$

whenever $\frac{1}{n} + 2\epsilon < \delta$. Then,

$$H^d_\delta([0,1]) \leq \lim_{n\to\infty} \lim_{\epsilon\to 0^+} n(\frac{1}{n} + 2\epsilon)^d = \lim_{n\to\infty} \frac{n}{n^d} = 0.$$

So, we see that $H^d_\delta([0,1]) = 0$ for all $\delta > 0$. It follows that

$$H^d([0,1]) = \sup_{\delta>0} H^d_\delta([0,1]) = 0.$$

(c) *Comment:* There was an error in the approach set out by the problem asked on the original homework. What follows is an appropriate replacement.

Let $L \in \mathbb{R}$ be positive, and let k be a positive integer. Let

$$\Delta = \{\mathbf{v} \in \mathbb{R}^k \ : \ v_i \geq 0 \text{ and } \sum_{i=1}^k v_i = L\}.$$

Fix a $d \in \mathbb{R}$ with $0 < d < 1$. Consider the function

$$m : \Delta \to \mathbb{R}; \quad \mathbf{v} \mapsto \sum_{i=1}^k \mathbf{v}_i^d.$$

Let $\mathbf{e}_i \in \mathbb{R}^k$ be the vector with a 1 in the i-th position and all other entries zero. Then the global minima for the function m occur at the points $L\mathbf{e}_i \in \Delta$, and so $m(\mathbf{v}) \geq L^d$ for all $\mathbf{v} \in \Delta$.

Solution: The statement is clear when $k = 1$, since $\Delta = \{L\}$ contains only one point which must be the minimum of m. Furthermore, $L = L\mathbf{e}_1$ is the mimimum.

Now consider the case when $k = 2$. Here

$$\Delta = \{(t, L-t) \ : \ 0 \leq t \leq L\}.$$

We have

$$m(t, L-t) = t^d + (L-t)^d.$$

Since L is a constant, we think of this as a function of t. Observe that the image of the function is positive and takes the value L^d at the endpoints. Since the function is differentiable, the minimum is either attained at the endpoints or at a critical point. Observe that

$$\frac{d}{dt}[t^d + (L-t)^d] = \frac{1}{d}\big(t^{d-1} - (L-t)^{d-1}\big).$$

At a critical point, we must have $t^{d-1} - (L-t)^{d-1} = 0$, which implies $t = \frac{L}{2}$. At this value of t, we see that

$$m(t, L-t) = \frac{2L^d}{2^d}.$$

Since $0 < d < 1$, this is clearly larger than L^d, so we see the minima occur at $(L, 0)$ and $(0, L)$, where the value of he function is L^d.

Finally consider the case when $k > 2$. Suppose $\mathbf{v} \in \mathbb{R}^k$ is a global minimum for m and that $\mathbf{v}$ is not of the form stated in the problem. Then there are distinct indices i and j so that $\mathbf{v}_i \neq 0$ and $\mathbf{v}_j \neq 0$. Let $\ell = \mathbf{v}_i + \mathbf{v}_j$. From case when $k = 2$, observe that

$$\ell^d < \mathbf{v}_i^d + \mathbf{v}_j^d.$$

Now define the vector $\mathbf{w} \in \mathbb{R}^k$ by

$$\mathbf{w}_n = \begin{cases} \mathbf{w}_n = \ell & \text{if } n = i \\ \mathbf{w}_n = 0 & \text{if } n = j \\ \mathbf{w}_n = \mathbf{v}_n & \text{otherwise.} \end{cases}$$

We observe that $\mathbf{w} \in \Delta$ because the sum of the entries are the same as the sum of the entries of $\mathbf{v}$ and

$$m(\mathbf{v}) - m(\mathbf{w}) = \mathbf{v}_i^d + \mathbf{v}_j^d - \ell^d > 0,$$

which contradicts our original assumption that $\mathbf{v}$ was a global minimum.

(d) Use the prior part to argue that $H^d([0,1]) = \infty$ whenever $0 < d < 1$.

Solution: Suppose $0 < d < 1$, and let $\delta > 0$. Let n be the largest number so that $2n\delta \leq 1$. We will show that

$$H_\delta^d([0,1]) \geq n\delta^d. \tag{1}$$

To prove the claim, let $\mathcal{I}$ be a countable covering of $[0,1]$ by open intervals of length less than δ. For $i \in \{1, \ldots, n\}$, let $A_i = [2i\delta, (2i+1)\delta]$. Define

$$\mathcal{I}_i = \{I \in \mathcal{I} \ : \ I \cap A_i \neq \emptyset\}.$$

Observe that the $\mathcal{I}_i$ are pairwise disjoint, because the intervals A_i are pairwise separated by at least δ. In particular,

$$\sum_{I \in \mathcal{I}} |I|^d \geq \sum_{i=1}^{n} \sum_{I \in \mathcal{I}_i} |I|^d.$$

Now observe that A_i is compact and $\mathcal{I}_i$ is an open cover. So, there is a finite subcover $\mathcal{J}_i \subset \mathcal{I}_i$. Then by the prior part, we see that

$$\sum_{I \in \mathcal{I}_i} |I|^d \geq \sum_{J \in \mathcal{J}_i} |J|^d \geq \delta^d.$$

Since the above inequality holds for all i, we see that

$$\sum_{I\in\mathcal{I}} |I|^d \geq n\delta^d.$$

Since $H^d_\delta([0,1])$ is an infimum of such quantities, we see equation 1 holds, i.e., $H^d_\delta([0,1]) \geq n\delta^d$.

In order to conclude the proof, we need to argue that $H^d_\delta([0,1])$ can be made arbitrarily large. Then it will follow from part (a) that $H^d([0,1]) = \infty$. For each $n \in \mathbb{N}$, let $\delta_n = \frac{1}{2n}$ (so that δ_n determines n as written above). Now using L'Hôpital's rule that

$$\lim_{n\to\infty} \frac{n}{(2n)^d} = \lim_{n\to\infty} \frac{1}{2(2n)^{d-1}} = \lim_{n\to\infty} \frac{(2n)^{1-d}}{2} = \infty.$$

Since $H^d_{\delta_n}([0,1]) \geq \frac{n}{(2n)^d}$, we see that $H^d_{\delta_n}([0,1])$ can be made arbitrarily large by taking $\delta = \frac{1}{2n}$ sufficiently small. Thus $H^d([0,1]) = \infty$ as claimed.

Final remarks on this problem: It can be shown that for any set $A \subset \mathbb{R}$, there is a unique $0 \leq D \leq 1$ so that

$$H^d(A) = \infty \quad \text{for } 0 \leq d < D \qquad \text{and} \qquad H^d(A) = 0 \quad \text{for } d > D.$$

This number D is called the *Hausdorff dimension* of A. Once you do the exercises above, you will have shown that the Hausdorff dimension of $[0,1]$ is 1.

If you would like a challenge, try to show that the Hausdorff dimension of the middle third Cantor set is $\frac{\log 2}{\log 3}$.