Math 70100: Functions of a Real Variable I
Homework 8, due Wednesday, November 5.

1. *(Combines Pugh's Ch. 6 # 1-2)* Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. Prove that $m^* \circ f(A) = |a| \cdot m^*(A)$ for each $A \subset \mathbb{R}$, where m^* is the Lebesgue outer measure on $\mathbb{R}$.

2. Use the formula from the prior problem to show that the middle third Cantor set C satisfies $m^*(C) = 0$, where m^* is Lebesgue outer measure. (*Hint:* Use the self-similarity.)

3. *(Royden §2.2 # 7)* A set of real numbers is said to be a G_δ set if it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which $E \subset G$ and $m^*(G) = m^*(E)$.

4. Fix some real number $d \geq 0$. For a subset $A \subset \mathbb{R}$ and $\delta > 0$, let

$$H^d_\delta(A) = \inf \Big\{ \sum_k |I_k|^d \Big\},$$

where the infimum is taken over all countable covers $\{I_k\}$ of A by open intervals each of which has length less than δ. The *d-dimensional Hausdorff outer measure* of A is

$$H^d(A) = \lim_{\delta \to 0} H^d_\delta(A).$$

You can use without proof that H^d is an outer measure. You may also use without proof that when $d = 1$, H^d is the Lebesgue outer measure on $\mathbb{R}$.

(a) Explain why if $\delta < \delta'$, then $H^d_\delta(A) \geq H^d_{\delta'}(A)$ for every $A \subset \mathbb{R}$. (*Remark*: It follows that $H^d(A) = \sup_{\delta > 0} H^d_\delta(A)$.)

(b) Show that $H^d([0, 1]) = 0$ for every $d > 1$.

(c) Let $L \in \mathbb{R}$ be positive, and let k be a positive integer. Let

$$\Delta = \{\mathbf{v} \in \mathbb{R}^k \ : \ v_i \geq 0 \text{ and } \sum_{i=1}^k v_i = L\}.$$

Fix a $d \in \mathbb{R}$ with $0 < d < 1$. Consider the function

$$m : \Delta \to \mathbb{R}; \quad \mathbf{v} \mapsto \sum_{i=1}^k v_i^d.$$

Show that this function has a unique global maximum which is attained at the vector where each $v_i = \frac{L}{k}$.

(d) Use the prior part to argue that $H^d([0, 1]) = \infty$ whenever $0 < d < 1$.

Final remarks on this problem: It can be shown that for any set $A \subset \mathbb{R}$, there is a unique $0 \leq D \leq 1$ so that

$$H^d(A) = \infty \quad \text{for } 0 \leq d < D \qquad \text{and} \qquad H^d(A) = 0 \quad \text{for } d > D.$$

This number D is called the *Hausdorff dimension* of A. Once you do the exercises above, you will have shown that the Hausdorff dimension of $[0, 1]$ is 1.

If you would like a challenge, try to show that the Hausdorff dimension of the middle third Cantor set is $\frac{\log 2}{\log 3}$.