Math 70100: Functions of a Real Variable I
Homework 7, due Wednesday, October 22.

1. Assume that $f_n : [0,1] \to \mathbb{R}$ is a sequence of differentiable functions whose derivatives are uniformly bounded. Suppose there is an $x_0 \in [0,1]$ so that $\{f_n(x_0) \ : \ n \in \mathbb{N}\}$ is bounded. Prove that $\{f_n\}$ has a subsequence which converges uniformly to a continuous function on $[0,1]$.

Solution: Let $\{f_n\}$ be a sequence of differentiable functions from $[0,1]$ to $\mathbb{R}$, and assume there is an $M > 0$ so that $|f_n'(x)| < M$ for all $x \in [0,1]$ and that $|f_n(x_0)| < M$ for some $x_0 \in [0,1]$.

By Arzelá-Ascoli, it suffices to prove that $\{f_n\}$ is pointwise totally bounded and equicontinuous.

First we claim that $\{f_n\}$ is equicontinuous. Let $\epsilon > 0$. We claim that $|x - y| < \frac{\epsilon}{M}$ implies $|f_n(x) - f_n(y)| < \epsilon$ for each $n \in \mathbb{N}$, which will verify the definition of equicontinuity. Suppose not. Then there is an n and an $x, y \in [0,1]$ so that $|x - y| < \frac{\epsilon}{M}$ but $|f_n(x) - f_n(y)| \geq \epsilon$. By the mean value theorem, there is a z between x and y so that

$$|f_n'(z)| = \frac{|f_n(x) - f_n(y)|}{|x - y|} > \frac{\epsilon}{\epsilon/M} = M.$$

But this contradicts our uniform bound on the derivative. Thus, we have shown that $\{f_n\}$ is equicontinuous.

Now we claim that $\{f_n\}$ is pointwise totally bounded. Recall that in $\mathbb{R}$, totally bounded is the same as bounded. We claim that for each $x \in [0,1]$, we have $|f_n(x)| < 2M$, which will verify by definition that $\{f_n\}$ is pointwise bounded. Suppose to the contrary that $|f_n(x)| \geq 2M$. Recall that $|f_n(x_0)| < M$. Therefore,

$$|f_n(x) - f_n(x_0)| > M.$$

But, then because $x, x_0 \in [0,1]$, we have $|x - x_0| \leq 1$. Again by the mean value theorem, there is a y between x and x_0 so that

$$|f'(z)| = \frac{|f_n(x) - f_n(x_0)|}{|x - x_0|} > \frac{M}{1}.$$

Again, this contradicts our uniform bound on the derivative.

2. *(Royden-Fitzpatrick §10.1 # 5)* A function $f : [0,1] \to \mathbb{R}$ is said to be Hölder continuous of order α provided there is a constant C for which

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad \text{for all} \quad x, y \in [0,1].$$

Define the Hölder norm

$$\|f\|_\alpha = \max \ \{|f(x)| + \frac{|f(x) - f(y)|}{|x - y|^\alpha} \ : \ x, y \in [0,1] \text{ and } x \neq y\}.$$

Show that for $0 < \alpha < 1$, the set of functions for which $\|f\|_\alpha \le 1$ has compact closure as a subset of subset of the space of continuous real-valued functions on $[0,1]$ with the uniform norm.

Solution: Fix some $\alpha \in (0,1)$. By the Arzelá-Ascoli theorem, it suffices to show that the set $\mathcal{F}$ of functions $f : [0,1] \to \mathbb{R}$ with $\|f\|_\alpha \le 1$ is pointwise totally bounded and equicontinuous.

Since the functions $\mathcal{F}$ map to $\mathbb{R}$, it suffices to show the functions are pointwise bounded. We claim that $|f(x)| \le 1$ for each $x \in [0,1]$ and each $f \in \mathcal{F}$. Observe that by definition of the Hölder norm, for each $x \in [0,1]$ and each $f \in \mathcal{F}$, we have

$$|f(x)| \le \|f\|_\alpha \le 1,$$

so 1 serves as a uniform pointwise bound.

We claim that $\mathcal{F}$ is equicontinuous. Fix an $\epsilon > 0$ an $x \in [0,1]$. We claim that for each $\epsilon > 0$ and each y with $|x - y| < \epsilon$ and each $f \in \mathcal{F}$, we have

$$|f(x) - f(y)| < \epsilon^\alpha.$$

Fix such an ϵ, y and f. If $x = y$, then it is trivially true. Otherwise, observe by definition of the Hölder norm, we have

$$\frac{|f(x) - f(y)|}{|x-y|^\alpha} \le 1.$$

Thus,

$$|f(x) - f(y)| \le |x-y|^\alpha < \epsilon^\alpha,$$

as claimed.

3. *(Lang §III.4 #21)* Let X be a metric space and E be a normed vector space. Let $BC(X,E)$ be the space of bounded continuous maps $X \to E$ (with the uniform norm). Let Φ be a bounded subset of $BC(X,E)$. For $x \in X$, let $\mathrm{ev}_x : \Phi \to E$ be the function $\mathrm{ev}_x(\phi) = \phi(x)$. Show that ev_x is continuous and bounded. Show that Φ is equicontinuous at a point $a \in X$ if and only if the map $x \mapsto \mathrm{ev}_x$ of X into $BC(\Phi, E)$ is continuous at a.

Solution: Fix Φ to be a bounded subset of $BC(X,E)$ as stated. We let $\|\cdot\|$ be the uniform norms on both $BC(X,E)$ and $BC(\Phi,E)$, and let $|\cdot|$ be the norm on E. The statement that Φ is bounded then gives an $M > 0$ so that $\phi(x) < M$ for each $\phi \in \Phi$ and each $x \in X$.

Consider the map $\mathrm{ev}_x : \Phi \to E$ as described in the problem. We claim this map is continuous and bounded. It is clearly bounded since $\phi(x) < M$ for each $\phi \in \Phi$. This is a map between metric spaces, so we will use the ϵ-δ definition of continuity to show ev_x is continuous at each point $\phi \in \Phi$. Fix some $\epsilon > 0$. We claim that for each $\psi \in \Phi$, we have $\|\phi - \psi\| < \epsilon$ implies $|\mathrm{ev}_x(\phi) - \mathrm{ev}_x(\psi)| < \epsilon$. Indeed, suppose $\|\phi - \psi\| < \epsilon$. Then by definition of the uniform norm, $|\phi(x) - \psi(x)| < \epsilon$. But this is the same as the statement that $|\mathrm{ev}_x(\phi) - \mathrm{ev}_x(\psi)| < \epsilon$ by definition of ev_x.

Now we will show that ev_x is continuous and bounded. Show that Φ is equicontinuous at a point $a \in X$ if and only if the map $x \mapsto \mathrm{ev}_x$ of X into $BC(\Phi, E)$ is continuous at a. Fix $a \in X$. Observe that the following statements are all equivalent. (Justifications for each step are given in parenthesis.)

- The map $x \mapsto \mathrm{ev}_x$ of X into $BC(\Phi, E)$ is continuous at a.
- For each $\epsilon > 0$, there is a $\delta > 0$ so that $d(x, a) < \delta$ implies $\|\mathrm{ev}_x - \mathrm{ev}_a\| < \epsilon$. (*Metric definition of continuity.*)
- For each $\epsilon > 0$, there is a $\delta > 0$ so that $d(x, a) < \delta$ and $\phi \in \Phi$ implies $|\mathrm{ev}_x(\phi) - \mathrm{ev}_a(\phi)| < \epsilon$. (*Definition of uniform norm.*)
- For each $\epsilon > 0$, there is a $\delta > 0$ so that $d(x, a) < \delta$ and $\phi \in \Phi$ implies $|\phi(x) - \phi(a)| < \epsilon$. (*Definition of* ev_**.*)
- Φ is equicontinuous at point a. (*Definition of equicontinuity.*)

4. *(Rudin's Principles of real analysis, Chapter 7 # 20)* Prove that if $f : [0, 1] \to \mathbb{R}$ is continuous and if
$$\int_0^1 f(x)x^n \, dx = 0$$
for all integers $n \geq 0$, then f is identically zero on $[0, 1]$. (*Hint:* This is a standard application of the Stone-Weierstrass Theorem or even just Weierstrass's theorem.)

Solution: Suppose to the contrary that f is not identically zero but that $\int_0^1 f(x)x^n \, dx = 0$ for every integer $n \geq 0$. Let
$$I = \int_0^1 f(x)^2 \, dx,$$
which is positive since f is not identically zero. Choose $M > 0$ so that $|f(x)| < M$ for each $x \in [0, 1]$, which exists by compactness of $[0, 1]$. Recall by Weierstrass's theorem that polynomials are uniformly dense in $C([0, 1])$, the space of continuous functions from $[0, 1]$ to $\mathbb{R}$. Thus, there is a polynomial $p(x)$ so that
$$|f(x) - p(x)| < \frac{I}{2M} \quad \text{for all } x \in [0, 1].$$
Then we see that
$$\left| \int_0^1 f(x)\big(f(x) - p(x)\big) \, dx \right| \leq \int_0^1 |f(x)|\big|f(x) - p(x)\big| \, dx \leq \int_0^1 M \frac{I}{2M} \, dx = \frac{I}{2}. \tag{1}$$
On the other hand, by linearity of the integral, we see
$$\int_0^1 f(x)p(x) \, dx = 0.$$

So, we compute

$$\left|\int_0^1 f(x)\big(f(x)-p(x)\big)\ dx\right| = \left|\int_0^1 f(x)^2\ dx - \int_0^1 f(x)p(x)\ dx\right| = |I-0| = I.$$

But since $I \neq 0$, this contradicts equation (1).

5. *(Kriz and Pultr §9.7 # 8)* Prove that any open set in $\mathbb{R}^n$ is σ-compact.

Solution: Let $U \subset \mathbb{R}^n$ be an open set. For $\mathbf{x} \in \mathbb{R}^n$, let $B_\epsilon(\mathbf{x}) \subset \mathbb{R}^n$ denote the open ball of radius ϵ centered at $\mathbf{x}$. For each integer $m \geq 1$, define the subset

$$K_m = \{\mathbf{x} \in U \ : \ |\mathbf{x}| \leq m \text{ and } B_{1/m}(\mathbf{x}) \subset U\}.$$

We claim K_m is compact. Since $K_m \subset \mathbb{R}^n$, it suffices to prove that K_m is a closed and bounded subset of $\mathbb{R}^n$. Clearly K_m is a bounded set, since $|\mathbf{x}| \leq m$. We claim it is also closed. Suppose $\mathbf{x}_k \in K_m$ converges to $\mathbf{x} \in \mathbb{R}^m$. We claim $\mathbf{x} \in K_m$. Observe $|\mathbf{x}_k| \leq m$ for each k, and by continuity of $|\cdot|$, we have

$$|\mathbf{x}| = \lim_{k\to\infty} |\mathbf{x}_k| \leq m.$$

It remains to show $B_{1/m}(\mathbf{x}) \subset U$. Choose a $\mathbf{y} \in B_{1/m}(\mathbf{x})$. Again by continuity of $|\cdot|$, we have

$$\frac{1}{m} > |\mathbf{x}-\mathbf{y}| = \lim_{k\to\infty} |\mathbf{x}_k - \mathbf{y}|.$$

So, we can find a k so that $|\mathbf{x}_k - \mathbf{y}| < \frac{1}{m}$. For this k,

$$\mathbf{y} \in B_{1/m}(\mathbf{x}_k) \subset U.$$

Since $\mathbf{y} \in B_{1/m}(\mathbf{x})$ was arbitrary, this shows $B_{1/m}(\mathbf{x}) \subset U$. Thus $\mathbf{x} \in K_m$.

We have shown each K_m is compact. So, to show U is σ-compact, it suffices to show that $U = \bigcup_m K_m$. By definition $K_m \subset U$, so the union is also a subset of U. To see the opposite inclusion, choose an $\mathbf{x} \in U$. Since U is open, there is an ϵ so that $B_\epsilon(\mathbf{x}) \subset U$. Choose an integer M so that

$$M > \max\ \big(|\mathbf{x}|, \frac{1}{\epsilon}\big).$$

Then, $|\mathbf{x}| < M$ and $B_{1/M}(\mathbf{x}) \subset B_\epsilon(\mathbf{x}) \subset U$. So by definition, $\mathbf{x} \in K_M$.