Math 70100: Functions of a Real Variable I
Homework 7, due Wednesday, October 22.

1. Assume that $f_n : [0,1] \to \mathbb{R}$ is a sequence of differentiable functions whose derivatives are uniformly bounded. Suppose there is an $x_0 \in [0,1]$ so that $\{f_n(x_0) \ : \ n \in \mathbb{N}\}$ is bounded. Prove that $\{f_n\}$ has a subsequence which converges uniformly to a continuous function on $[0,1]$.

2. *(Royden-Fitzpatrick §10.1 # 5)* A function $f : [0,1] \to \mathbb{R}$ is said to be Hölder continuous of order α provided there is a constant C for which
$$|f(x) - f(y)| \leq C|x-y|^\alpha \quad \text{for all} x, y \ in [0,1].$$
Define the Hölder norm
$$\|f\|_\alpha = \max \ \{|f(x)| + \frac{|f(x)-f(y)|}{|x-y|^\alpha} \ : \ x, y \in [0,1] \text{ and } x \neq y\}.$$
Show that for $0 < \alpha < 1$, the set of functions for which $\|f\|_\alpha \leq 1$ has compact closure as a subset of subset of the space of continuous real-valued functions on $[0,1]$ with the uniform norm.

3. *(Lang §III.4 #21)* Let X be a metric space and E be a normed vector space. Let $BC(X,E)$ be the space of bounded continuous maps $X \to E$ (with the uniform norm). Let Φ be a bounded subset of $BC(X,E)$. For $x \in X$, let $\mathrm{ev}_x : \Phi \to E$ be the function $\mathrm{ev}_x(\phi) = \phi(x)$. Show that ev_x is continuous and bounded. Show that Φ is equicontinuous at a point $a \in X$ if and only if the map $x \mapsto \mathrm{ev}_x$ of X into $BC(\Phi, E)$ is continuous at a.

4. *(Rudin's Principles of real analysis, Chapter 7 # 20)* Prove that if $f : [0,1] \to \mathbb{R}$ is continuous and if
$$\int_0^1 f(x)x^n \, dx = 0$$
for all integers $n \geq 0$, then f is identically zero on $[0,1]$. (*Hint:* This is a standard application of the Stone-Weierstrass Theorem or even just Weierstrass's theorem.)

5. *(Kriz and Pultr §9.7 # 8)* Prove that any open set in $\mathbb{R}^n$ is σ-compact.