Math 70100: Functions of a Real Variable I
Homework 6, due Wednesday, October 15.

1. *(Royden-Fitzpatrick §10.2 #)* A point p in a topological space is called *isolated* if $\{p\}$ is open. Show that a complete metric space without isolated points is uncountable. (*Hint:* Use Baire's Theorem.)

Solution: Suppose to the contrary that X is a compete metric space which is countable and has no isolated points. By countability, we can write $X = \bigcup_{i \in I} \{x_i\}$ for some countable indexing set I. Observe that each $\{x_i\}$ is closed because we are in a metric space. Since X is complete and we have written X as a countable union of closed sets, the Baire Category theorem guarantees that some $\{x_i\}$ must have interior. That is, there is an $i \in I$ and a non-empty open subset $A \subset \{x_i\}$. But since A is non-empty, we must have $A = \{x_i\}$ and we conclude that $\{x_i\}$ is open. But then x_i is isolated, which is a contradiction.

2. Prove the following variant of the Baire Category theorem:
Suppose that X is a locally-compact Hausdorff space. Prove that if $X = \bigcup_{n=1}^{\infty} C_n$, where each C_n is closed, then some C_n has non-empty interior.
(*Hint:* Recall that a compact Hausdorff space is normal. Mimic the proof of the Baire Category theorem using open sets with compact closure obtained by normality rather than balls. You will need to use the finite intersection property, which characterizes compact sets.)

Solution: Suppose X is a locally-compact Hausdorff space. Let $\{C_n : n \in \mathbb{N}\}$ be a collection of closed sets. Assume no C_n has interior. We will show that $X \neq \bigcup_{n=1}^{\infty} C_n$.

We will need the following:
Claim: If U is non-empty open, then for any $n \in \mathbb{N}$ there is a non-empty open set V with compact closure $\bar{V} \subset U \setminus C_n$.
Proof of claim: Fix $n \in \mathbb{N}$ and U open. Since C_n has no interior, we know $U \not\subset C_n$. So, we can find an $x \in U \setminus C_n$. Now choose a compact neighborhood K of x. Since K is Hausdorff and compact, it is normal. The set $\{x\}$ is closed because X is Hausdorff. Observe that the closed sets ∂K, $K \setminus U$ and $K \cap C_n$ are all disjoint from $\{x\}$, so by normality, we can find an open subset $V \subset K$ containing x whose closure is disjoint from ∂K, $K \setminus U$ and C_n. In particular, then

$$\bar{V} \subset (K \cap U) \setminus C_n,$$

and $\bar{V}$ is compact because it is a closed subset of K. This proves the claim.

Now we inductively define a sequence of open sets with compact closures $\bar{V}_n$ so that $\bar{V}_n \subset V_{n-1}$ for each $n \geq 1$, and so that each $\bar{V}_n$ is disjoint from C_n. To do this, first observe that X is open, so by the claim there is an open V_1 with compact closure $\bar{V}_1$ so that $\bar{V}_1 \cap C_1 = \emptyset$. Now suppose that V_n is defined. Then, by the claim there is an open set V_{n+1} so that $\bar{V}_{n+1} \subset V_n \setminus C_{n+1}$.

We have defined a nested sequence of compact sets $\{\bar{V}_n\}$. We claim the intersection is non-empty. Observe that because these sets are nested, they satisfy the finite intersection property. (Concretely,

$$\bigcap_{i=1}^{k} \bar{V}_{n_k} \supset \bar{V}_{\max\{n_1,\ldots,n_k\}}.)$$

Then, by compactness of $\bar{V}_1$, the intersection $\bigcap_{n\in\mathbb{N}} \bar{V}_n$ is non-empty. (This is the property that any collection of closed sets in a compact set with the finite intersection property has the "grand" intersection property; see Proposition 3.1 of Lang's chapter II.) Let x be a point in this intersection. Observe that each V_n is disjoint from C_n, so $x \notin \bigcup_n C_n$. We conclude $\bigcup_n C_n \neq X$ as desired.

3. *(based on Lang III§4 #9)* We say a sequence $\{f_n\}$ of real valued functions on a topological space X is *monotone increasing* if for each $x \in X$ and each $n \in \mathbb{N}$, $f_{n+1}(x) \geq f_n(x)$. Recall $\{f_n\}$ converges to f pointwise if for each $x \in X$, we have $\lim_{n\to\infty} f_n(x) = f(x)$.

   Prove Dini's theorem:
   If $\{f_n\}$ is a monotone increasing sequence of continuous real-valued functions on a compact set metric space X which converges pointwise to a continuous function $f : X \to \mathbb{R}$, then the sequence converges uniformly.

Solution: **Solution 1.** Fix some $\epsilon > 0$. We will show that there is an N so that $n > N$ implies

$$\sup_{x\in X} |f(x) - f_n(x)| < \epsilon,$$

which verifies the definition of uniform convergence. Note by continuity and compactness, the supremum is realized, so it suffices to show that $|f(x) - f_n(x)| < \epsilon$ for every $x \in X$ and $n > N$.

Set $g_n(x) = f(x) - f_n(x)$. Observe that g_n is continuous, non-negative, and for each x, the sequence $\{g_n(x) \ : \ n \in \mathbb{N}\}$ decreases monotonically and converges to zero. Set

$$U_n = \{x \in X \ : \ g_n(x) < \epsilon\}.$$

Then because $g_{n+1}(x) \leq g_n(x)$ for all $x \in X$, we have $U_n \subset U_{n+1}$. Also because $U_n = g_n^{-1}\big((-\infty, \epsilon)\big)$, we know U_n is open. Furthermore, for each x, we know $g_n(x) \to 0$ as $n \to \infty$, so each x lies in some U_n. That is, $\bigcup_n U_n = X$. Then, because X is compact, there is a finite list $\{n_1, \ldots, n_k\}$ so that

$$X = \bigcup_{i=1}^{k} U_{n_i}.$$

Let $N = \max\{n_1, \ldots, n_k\}$. Then $U_{n_i} \subset U_N$ for each $i \in \{1, \ldots, k\}$, and we conclude that $X = U_N$. Then if $n > N$, for every $x \in X$ we have $x \in U_N$ and hence

$$|f(x) - f_n(x)| = g_n(x) \leq g_N(x) < \epsilon.$$

This concludes the proof by the remarks in the first paragraph of the proof.

Solution 2. We utilize Arzelà-Ascoli to obtain the uniform convergence. The sequence $\{f_n\}$ is pointwise bounded because each $x \in X$ and each $n \in \mathbb{N}$ satisfies

$$f_1(x) \le f_n(x) \le f(x).$$

Now we will show that f_n is equicontinuous at each $x \in X$. Pick $x \in X$ and $\epsilon > 0$. We must show that there is a δ so that $d(x, y) < \delta$ and $n \in \mathbb{N}$ implies $|f_n(y) - f_n(x)| < \epsilon$. Since x is fixed and $f_n(x) \to f(x)$ monotonically from below as $n \to \infty$, there is an $N \in \mathbb{N}$ so that

$$|f(x) - f_N(x)| = f(x) - f_N(x) < \frac{\epsilon}{2}.$$

Since $\{f\} \cup \{f_i \;:\; i \le N\}$ is a finite collection of continuous functions, there is a $\delta > 0$ so that for each $y \in X$ with $d(x, y) < \delta$ and each $i \le N$, we have

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \text{and} \quad |f_i(x) - f_i(y)| < \frac{\epsilon}{2}.$$

We claim that $d(x, y) < \delta$ and $n \in \mathbb{N}$ implies $|f_n(y) - f_n(x)| < \epsilon$. This is true by definition of δ if $n \le N$. Now assume $n > N$ and $d(x, y) < \delta$. Then,

$$f_N(x) \le f_n(x) \le f(x) \quad \text{and} \quad f_N(y) \le f_n(y) \le f(y).$$

It follows that

$$f_N(y) - f(x) \le f_n(y) - f_n(x) \le f(y) - f_N(x). \tag{1}$$

Observe that:

$$|f(y) - f_N(x)| \le |f(y) - f(x)| + |f(x) - f_N(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$$|f_N(y) - f(x)| \le |f_N(y) - f_N(x)| + |f_N(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Then by equation 1, we have

$$|f_n(y) - f_n(x)| \le \max\big(|f_N(y) - f(x)|, |f(y) - f_N(x)|\big) < \epsilon,$$

verifying equicontinuity.

Since $\{f_n\}$ is pointwise bounded and equicontinuous, the Arzelà-Ascoli theorem tells us that there is a subsequence f_{n_j} which converges uniformly. Since this subsequence converges pointwise to f, the uniform limit of f_{n_j} must be f. Now we claim that $f_n \to f$ uniformly. This essentially follows from the fact that for each $x \in X$, $n < m$ implies $f_n(x) \le f_m(x) \le f(x)$. In particular, $n < m$ implies

$$\|f_m - f\| \le \|f_n - f\|,$$

where $\|\cdot\|$ denotes the uniform norm. Therefore, because $\|f_{n_j} - f\| \to 0$ as $j \to \infty$, we have $\|f_n - f\| \to 0$ as $n \to \infty$, which concludes the proof. (*Explanation:* To see $\|f_n - f\| \to 0$, let $\epsilon > 0$. Since $\|f_{n_j} - f\| \to 0$, there is a J so that $j \ge J$ implies $\|f_{n_j} - f\| < \epsilon$. Then for $n > n_J$, we have $\|f_n - f\| \le \|f_{n_J} - f\| \le \epsilon$.)