Math 70100: Functions of a Real Variable I
Homework 6, due Wednesday, October 15.

1. *(Royden-Fitzpatrick §10.2 #)* A point p in a topological space is called *isolated* if $\{p\}$ is open. Show that a complete metric space without isolated points is uncountable. (*Hint:* Use Baire's Theorem.)

2. Prove the following variant of the Baire Category theorem:
   Suppose that X is a locally-compact Hausdorff space. Prove that if $X = \bigcup_{n=1}^{\infty} C_n$, where each C_n is closed, then some C_n has non-empty interior.
   (*Hint:* Recall that a compact Hausdorff space is normal. Mimic the proof of the Baire Category theorem using open sets with compact closure obtained by normality rather than balls. You will need to use the finite intersection property, which characterizes compact sets.)

3. *(based on Lang III§4 #9)* We say a sequence $\{f_n\}$ of real valued functions on a topological space X is *monotone increasing* if for each $x \in X$ and each $n \in \mathbb{N}$, $f_{n+1}(x) \geq f_n(x)$. Recall $\{f_n\}$ converges to f pointwise if for each $x \in X$, we have $\lim_{n\to\infty} f_n(x) = f(x)$.

   Prove Dini's theorem:
   If $\{f_n\}$ is a monotone increasing sequence of continuous real-valued functions on a compact set metric space X which converges pointwise to a continuous function $f : X \to \mathbb{R}$, then the sequence converges uniformly.