Math 70100: Functions of a Real Variable I
Homework 5, due Wednesday, October 8.

Name: Insert your name here.

1. *(Folland 4.7.68)* For a space X, let $C(X)$ denote the continuous functions from X to $\mathbb{R}$ equipped with the uniform norm. Let X and Y be compact Hausdorff spaces. Show that the algebra generated by functions of the form $f(x, y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$ is dense inside of $C(X \times Y)$.

Solution: Let $\mathcal{A}$ denote the algebra of functions on $X \times Y$ generated by functions of the form

$$g \cdot h : (x, y) \mapsto g(x)h(y),$$

where $g \in C(X)$ and $h \in C(Y)$. To show $\mathcal{A}$ is dense in $C(X \times Y)$, we will apply the Stone-Weierstrass theorem. We will now check the hypotheses for this theorem. First observe that the product of two compact spaces is compact. Second, observe that $\mathcal{A}$ is vanishes nowhere because product of two non-zero constant functions lies in $\mathcal{A}$. Now we will show that $\mathcal{A}$ separates points. Let $(x_1, y_1), \in (x_2, y_2)$ be distinct. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality, assume $x_1 \neq x_2$. Since X is Hausdorff, the sets $\{x_1\}$ and $\{x_2\}$ are closed. Recall that because X is Hausdorff and compact, it is normal. So, by Urysohn's Lemma, we can find a continuous function $g : X \to \mathbb{R}$ so that $g(x_1) = 0$ and $g(x_2) = 1$. Let $h : Y \to \mathbb{R}$ be the constant function $h(y) = 1$. Then,

$$g \cdot h(x_1, y_1) = g(x_1)h(y_1) = 0 \cdot 1 = 0 \quad \text{and} \quad g \cdot h(x_2, y_2) = g(x_2)h(y_2) = 1 \cdot 1 = 1,$$

showing that $\mathcal{A}$ separates points. Since $X \times Y$ is compact and $\mathcal{A}$ separates points and vanishes nowhere, by the Stone-Weierstrass theorem, $\mathcal{A}$ is dense in $X \times Y$.

2. *(Folland 4.7.69)* Let A be a nonempty set, and let $X = [0, 1]^A$. Show that the algebra generated by the coordinate maps $\pi_a : X \to [0, 1]$ and the constant function $\mathbf{1}$ is dense in $C(X)$.

Solution: Recall that X is compact by Tychonoff's theorem. Let $\mathcal{A}$ denote the algebra generated by the maps of the form π_a and the constant function $\mathbf{1}$. Observe that the algebra vanishes nowhere because $\mathbf{1}$ never takes the value zero. Now we will show $\mathcal{A}$ separates points. Let x and y be distinct points in X. Then (by definition of equality in $[0, 1]^A$, there is some a so that $\pi_a(x) \neq \pi_a(y)$. Since $\pi_a \in \mathcal{A}$ and $\pi_a(x) \neq \pi_a(y)$, we see $\mathcal{A}$ separates points. We have checked all the hypotheses of the Stone-Weierstrass theorem and conclude that $\mathcal{A}$ is dense in $C(X)$.

3. *(Rephrased Lang III §4 # 19)* Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \ : \ x \geq 0\}$, and let $C_0(\mathbb{R}_{\geq 0})$ denote the continuous real-valued functions on $\mathbb{R}_{\geq 0}$ which vanish at infinity. Prove that $C_0(\mathbb{R}_{\geq 0})$ is the uniform closure of the collection of all functions of the form $e^{-x}p(x)$, where p is a polynomial. (*Lang's Hint; note he phrases this question differently:* First show that you can approximate e^{-2x} by $e^{-x}q(x)$ for some polynomial q(x), by using Taylor's formula with remainder. If p is a polynomial, approximate $e^{-nx}p(x)$ by $e^{-x}q(x)$ for some polynomial q.)

Solution: For notational convenience, we let $\phi(x) = e^{-x}$. We do this so we can write expressions like $p\phi$ to denote the function $x \mapsto p(x)e^{-x}$.

Define the collection S of functions on $\mathbb{R}_{\geq 0} = (0, +\infty)$ by

$$S = \{p \cdot \phi \ : \ p \text{ is a polynomial}\}.$$

Let $\bar{S}$ denote the closure of $S \subset C(\mathbb{R}_+)$ in the uniform topology.

We write ϕ^2 for the function $x \mapsto \phi(x)^2 = e^{-2x}$. Following Lang's hint, we make the following claim:

Claim 1. The function ϕ^2 lies in $\bar{S}$. That is, for every $\epsilon > 0$, there is an polynomial p so that $\|\phi^2 - p\phi\| < \epsilon$, where $\|\cdot\|$ denotes the uniform norm.

Taylor's theorem gives us polynomial approximations to e^{-x}. Namely, we will use the Taylor polynomials centered at zero:

$$p_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^k.$$

The Lagrange form of the remainder for Taylor's theorem tells us that

$$|e^{-x} - p_n(x)| < \frac{f^{k+1}(a)}{(n+1)!} x^{n+1} \quad \text{for some } a \text{ with } 0 < a < x.$$

Observe that $|f^{k+1}(a)| = e^{-a}$, so this is bounded by 1 independently of k. That is,

$$|e^{-x} - p_n(x)| < \frac{1}{(n+1)!} x^{n+1}.$$

Therefore, we have

$$\|\phi^2 - p_n\phi\| = \sup_{x>0} |e^{-2x} - p_n(x)e^{-x}| = \sup_{x>0} e^{-x}|e^{-x} - p_n(x)| < \sup_{x>0} \frac{e^{-x}x^{n+1}}{(n+1)!}.$$

Observe that for any n, this supremum is realized because $e^{-x}x^n$ is zero at zero and tends to zero as $x \to +\infty$. By differentiability, this supremum must be realized at a critical point of the function $x \mapsto x^{n+1}e^{-x}$, whose derivative is $-x^n e^{-x}(x - n - 1)$. The only possible critical point is $x = n + 1$, so we see that the supremum is just the function evaluated at this point. In summary,

$$\|\phi^2 - p_n\phi\| < \frac{(n+1)^{(n+1)}}{e^{n+1}(n+1)!}.$$

Now we need to show that $\|\phi^2 - p_n\phi\| \to 0$ as $n \to \infty$. To do this we use Stering's formula:

$$\lim_{n\to\infty} \frac{e^n n!}{n^n\sqrt{2\pi n}} = 1.$$

By multiplying through along this sequence,

$$0 \leq \lim_{n\to\infty} \|\phi^2 - p_{n-1}\phi\| \leq \lim_{n\to\infty} \frac{n^n}{e^n n!} = \lim_{n\to\infty} \Big(\frac{n^n}{e^n n!}\Big)\Big(\frac{e^n n!}{n^n\sqrt{2\pi n}}\Big) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi n}} = 0.$$

So, we see $\|\phi^2 - p_{n-1}\phi\|$ tends to 0 as claimed. This completes the proof of Claim 1.

We will disregard the rest of Lang's hint, and solve the problem a slightly different way than he suggests.

Claim 2. We claim that $\bar{S}$ is an algebra. That is, we claim:

1. If $f \in \bar{S}$ and $c \in \mathbb{R}$, then $cf \in \bar{S}$.
2. If $f, g \in \bar{S}$, then $f + g \in \bar{S}$.
3. If $f, g \in \bar{S}$, then the product $fg : x \mapsto f(x)g(x)$ lies in $\bar{S}$.

We will prove each of these in the statements below. (The first two are fairly straightforward, and could probably be dismissed as "obvious" but we include them anyway.)

1. Let $f \in \bar{S}$ and $c \in \mathbb{R}$. Let $\epsilon > 0$. We will show that $cf \in \bar{S}$ by showing that there is a polynomial q so that $\|cf - q\phi\| < \epsilon$. Since $f \in \bar{S}$, we can find a polynomial p so that $\|f - p\phi\| < \frac{\epsilon}{|c|}$. Then consider the polynomial cp. We have

$$\|cf - cp\phi\| = |c| \cdot \|f - p\phi\| < \epsilon.$$

2. Let $f, g \in \bar{S}$. We claim that $f + g \in \bar{S}$. Fix some $\epsilon > 0$. Since both f and g lie in $\bar{S}$, there are polynomials p and q so that

$$\|f - p\phi\| < \frac{\epsilon}{2} \quad \text{and} \quad \|g - q\phi\| < \frac{\epsilon}{2}.$$

Then, $p + q$ is a polynomial and, by the triangle inequality,

$$\|f + g - (p + q)\phi\| = \|f - p\phi + g - q\phi\| \leq \|f - p\phi\| + \|g - q\phi\| < \epsilon.$$

3. Let $f, g \in \bar{S}$. We claim that $fg \in \bar{S}$. Fix some $\epsilon > 0$. Since $p \in \bar{S}$, there is a polynomial p so that

$$\|f - p\phi\| < \frac{\epsilon}{3\|g\|},$$

where we take $\frac{\epsilon}{3\|g\|} = +\infty$ if $\|g\| = 0$. Since $q \in \bar{S}$, there is a polynomial q so that

$$\|g - p\phi\| < \frac{\epsilon}{3\|p\phi\|},$$

where again we follow the convention that $\frac{\epsilon}{3\|pe^{-x}\|} = +\infty$ when $\|pe^{-x}\| = 0$. Also by Claim 1, there is a polynomial r so that

$$\|\phi^2 - r\phi\| < \frac{\epsilon}{3\|pq\|},$$

where we continue to follow convention. Observe pqr is a polynomial. Now using the above inequalities, and the triangle inequality, we have:

$$\begin{aligned} \|fg - pqr\phi\| &\leq \|fg - p\phi g\| + \|p\phi g - pq\phi^2\| + \|pq\phi^2 - pqr\phi\| \\ &= \|g\|\|f - p\phi\| + \|p\phi\|\|g - q\phi\| + \|pq\|\|\phi^2 - r\phi\| \\ &< \|g\|\frac{\epsilon}{3\|g\|} + \|p\phi\|\frac{\epsilon}{3\|p\phi\|} + \|pq\|\frac{\epsilon}{3\|pq\|} = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This concludes the proof of Claim 2.

Final argument. Claim 2 told us that $\bar{S}$ is an algebra, and it is of course closed. Also observe that the function $\phi(x) = e^{-x}$ lies in $\bar{S}$. This function separates points, because it is strictly monotone decreasing. Also this function takes only non-zero values, so it is nowhere vanishing. This means that the algebra $\bar{S}$ separates points and is nowhere vanishing. Observe that $\mathbb{R}_+$ is locally compact. So, by the locally compact version of the Stone-Weierstrass theorem, $\bar{S} = C_0(\mathbb{R}_{\geq 0})$ and S is dense in $\mathbb{R}_{\geq 0}$.