Math 70100: Functions of a Real Variable I
Homework 4, due Wednesday, October 1.

1. *(Lang Chapter 2, problem 20)* Recall that a space is called *second countable* if it has a countable base for its topology. (Lang calls this notion separable.) A topological space is *metrizable* if it has a metric which induces the same topology on the space. A space is *normal* if it is Hausdorff and for any two disjoint closed sets A and B there are open sets U and V with $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

   Prove that a normal separable space X is metrizable. (Follow the hint suggested by Lang.) (This is the *Urysohn Metrization Theorem*.)

Solution: We follow the hint of Lang. Suppose X is normal and separable. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable basis. Observe that the collection $\mathcal{P}$ of all pairs $(U_n, U_m) \in \mathcal{U} \times \mathcal{U}$ with $\bar{U}_n \subset U_m$ is countable (as it is a subset of the countable set $\mathcal{U} \times \mathcal{U}$.) Therefore, it can be enumerated as

$$\mathcal{P} = \{(U_{n(i)}, U_{m(i)}) \ : \ i \in \mathbb{N}\}.$$

For each $i \in \mathbb{N}$, Urysohn's lemma gives us a continuous function $f : X \to [0,1]$ so that $f(x) = 0$ for $x \in \bar{U}_{n(i)}$ and $f(x) = 1$ for $x \notin U_{m(i)}$. For $x, y \in X$, define

$$d(x,y) = \sum_{i \in \mathbb{N}} \frac{1}{2^i} |f_i(x) - f_i(y)|.$$

Observe that the series (sum) converges because it is bounded by the convergent series, $\sum_{i \in \mathbb{N}} \frac{1}{2^n} = 1$.

First we claim that d is a metric. Clearly $d(x,y) = d(y,x) \geq 0$ and $d(x,x) = 0$ for $x, y \in X$. Also, it satisfies the triangle inequality, because we can apply the triangle inequality on $\mathbb{R}$ term-wise:

$$d(x,y) + d(y,z) = \sum_{i \in \mathbb{N}} \frac{1}{2^i}\big(|f_i(x) - f_i(y)| + |f_i(y) - f_i(z)|\big) \geq \sum_{i \in \mathbb{N}} \frac{1}{2^i}|f_i(x) - f_i(z)| = d(x,z).$$

We still need to show that $x \neq y$ implies $d(x,y) > 0$. For this we use the following observation:

Claim. If $x \in V$ with V open in X, then there is an $i \in \mathbb{N}$ so that the $x \in U_{n(i)}$ and $U_{m(i)} \subset V$.

Proof of claim. Since $\mathcal{U}$ is a basis, there is a $U \in \mathcal{U}$ with $x \in U' \subset V$. Because X is Hausdorff, the set $\{x\}$ is closed. Thus $\{x\}$ and $X \smallsetminus U$ are disjoint and closed. By normality, there are disjoint open sets U' and V' with $x \in U'$ and $X \smallsetminus U \subset V'$. In particular, we see that $x \in U' \subset \bar{U}' \subset X \smallsetminus V'$. Again because $\mathcal{U}$ is a basis, we can find an $U'' \in \mathcal{U}$ with $x \in U'' \subset U'$. Now observe that we have found a U and U'' in $\mathcal{U}$ with $x \in U'' \subset \bar{U}'' \subset U \subset V$. By definition of $\mathcal{P}$, we see $(U'', U) \in \mathcal{P}$, so there is an $i \in \mathbb{N}$ so that $(U'', U) = (U_{n(i)}, U_{m(i)})$.

Now suppose $x \neq y$. We will show that $d(x,y) > 0$. Because X is normal it is Hausdorff. So, there are disjoint open sets U and V with $x \in U$ and $y \in V$. Now by the claim there is an i with $x \in U_{n(i)}$ and $U_{m(i)} \subset U$. Observe that by definition of f_i we have $f_i(x) = 0$ and $f_i(y) = 1$. Thus,

$$d(x, y) \geq \frac{1}{2^i}|f_i(x) - f_i(y)| = \frac{1}{2^i}.$$

Now we need to see that d induces the same topology. Let $\mathcal{T}$ denote the original topology, and $\mathcal{M}$ denote the metric topology induced by d. We will show that they have the same open sets.

Let U be open in $\mathcal{T}$ and let $x \in U$. Then by the claim, we can find an i so that $x \in U_{n(i)}$ and $U_{m(i)} \subset U$. Set $r = \frac{1}{2^i}$. We will show that $B_r(x) \subset U$, which implies that U is open in $\mathcal{M}$. Equivalently, we can show that $X \setminus U \subset X \setminus B_r(x)$. Suppose $y \notin U$. Because $U_{m(i)} \subset U$, we have $f_i(y) = 1$. Also because $x \in U_{n(i)}$, we have $f_i(x) = 0$. Thus,

$$d(x, y) \geq \frac{1}{2^i}|f_i(x) - f_i(y)| = r,$$

which means that $y \notin B_r(x)$ as desired.

Now choose $x \in X$ and $r > 0$. We will show the open ball $B_r(x)$ is open in $\mathcal{T}$. Since $r > 0$, there is an $N \in \mathbb{N}$ so that

$$\sum_{i=N}^{\infty} \frac{1}{2^i} < \frac{r}{2}.$$

Now define the function $\delta : X \to \mathbb{R}$ by

$$\delta(y) = \sum_{i=1}^{N-1} \frac{1}{2^i}|f_i(x) - f_i(y)|.$$

Since this is a finite sum of continuous functions, we know that δ is continuous. Let $U = \delta^{-1}\big((-\infty, \frac{r}{2})\big)$. By continuity of δ, U is open. Also observe that because $\delta(x) = 0$, we know $x \in U$. We claim that $U \subset B_r(x)$. Let $y \in U$. Then,

$$d(x, y) = \delta(y) + \sum_{i=N}^{\infty} \frac{1}{2^i}|f_i(x) - f_i(y)| \leq \delta(y) + \sum_{i=N}^{\infty} \frac{1}{2^i} < \frac{r}{2} + \frac{r}{2} = r.$$

So, $y \in B_r(x)$ as claimed.

2. *(Royden-Fitzpatrick §12.1 # 6)* Let X be a set and $\mathcal{T}$ be a topology on X. Let $C(X)$ denote the collection of all continuous real-valued functions on $(X, \mathcal{T})$, and let $\mathcal{W}$ denote the weak topology induced by $C(X)$. (That is $\mathcal{W}$ is the coarsest topology on X so that every $f \in C(X)$ is continuous.) Show that if $(X, \mathcal{T})$ is normal, then the two topologies are identical.

Solution: We show the topologies are identical by showing that they have the same open sets.

A subbasis for the weak topology, $\mathcal{W}$, is given sets of the form $f^{-1}(J)$ where $J \subset \mathbb{R}$ is open and $f \in C(X)$. Then by definition of $C(X)$, f is continuous and therefore $f^{-1}(J)$ is open in $(X, \mathcal{T})$. This shows that $\mathcal{W} \subset \mathcal{T}$.

Now let U be open in $(X, \mathcal{T})$. Let $x \in U$. Since $(X, \mathcal{T})$ is normal, it is Hausdorff, which implies that $\{x\}$ is closed. Observe that $\{x\}$ and $X \setminus \{U\}$ are closed and disjoint. So, Urysohn's lemma implies that there is a continuous function $f_x : X \to [0,1]$ so that $f(x) = 1$ and $f(y) = 0$ for each $y \in X \setminus U$. By definition of the weak topology, $W_x = f^{-1}\big((\frac{1}{2}, \infty)\big) \subset U$ is open in $\mathcal{W}$. We may make a choice of f_x and W_x for each $x \in U$. So, $U = \bigcup_{x \in U} W_x$ is open in $\mathcal{W}$.

3. *(Following Rudin's Real and Complex Analysis, pp. 69)* Let X be a locally compact Hausdorff space. (Recall this means that every $x \in X$ has a compact neighborhood.)

A *compactly supported function on* X is a function $f : X \to \mathbb{R}$ so that there is a compact set $K \subset X$ so that $f(x) = 0$ for $x \notin K$. We write $C_c(X)$ to denote the collection of all continuous compactly supported functions on X.

A function $f : X \to \mathbb{R}$ *vanishes at infinity* if for all $\epsilon > 0$ there is a compact set $K \subset X$ so that $|f(x)| < \epsilon$ for $x \notin K$. We write $C_0(X)$ to denote the collection of all continuous functions which vanish at ∞.

We endow these spaces with the uniform (or sup) norm. Observe that $C_c(X) \subset C_0(X)$.

(a) Show that $C_c(X)$ is dense in $C_0(X)$. (*Hint:* You need the version of Urysohn's lemma given in class: If X is locally compact and Hausdorff, and $K \subset U \subset X$ with K compact and U open, then there is a continuous $f : X \to [0,1]$ so that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \notin U$.)

Solution: Let $f \in C_0(X)$. It suffices to show that for each $\epsilon > 0$, there is a $g \in C_c(X)$ with $\|f - g\| < \epsilon$, where $\|\cdot\|$ denotes the uniform norm. Fix $\epsilon > 0$. Since $f \in C_0(X)$, there is a compact set $K \subset X$ so that $|f(x)| < \epsilon$ when $x \notin K$. Since X is locally compact, for each $x \in K$ we can find an open neighborhood $U_x \subset X$ of x with compact closure, $\bar{U}_x$. Then, $\{U_x : x \in K\}$ is an open cover of K, which by compactness has a finite subcover

$$\{U_{x_1}, \ldots, U_{x_n}\}.$$

Let $V = \bigcup_{i=1}^n U_{x_i}$. Then K and $X \setminus V$ are closed and disjoint, so Urysohn's lemma yields a continuous function $h : X \to [0,1]$ so that $h(x) = 1$ for $x \in K$ and $h(x) = 0$ for $x \notin V$. Consider the product function

$$h \cdot f : X \to \mathbb{R}; \quad x \mapsto h(x)f(x).$$

We see that $h \cdot f \in C_c(X)$ since h is zero outside the compact set $\bar{V} = \bigcup_{i=1}^n \bar{U}_{x_i}$. (Note a finite union of compact sets is compact.) Also, we have $\|f - h \cdot f\| < \epsilon$, because:

- If $x \in K$ then $f(x) - h(x)f(x) = f(x) - 1 \cdot f(x) = 0$.

- If $x \notin K$, then $0 \leq h(x) \leq 1$ and $|f(x)| < \epsilon$, so

$$|f(x) - h(x)f(x)| = \big|1 - h(x)\big||f(x)| \leq |f(x)| < \epsilon.$$

This concludes the argument: $g = h \cdot f \in C_c(X)$ satisfies $\|f - g\| < \epsilon$.

(b) Show that $C_0(X)$ is a Banach space (i.e., that it is complete).

Together, this shows that $C_0(X)$ is the metric completion of $C_c(X)$.

Solution: First observe that $C_0(X)$ is a subspace of the bounded continuous functions from X to $\mathbb{R}$. (An $f \in C_0(X)$ is bounded because there is a compact set K so that $|f(x)| < 1$ for $x \notin K$, and by continuity and compactness, f is bounded on K.)

Recall that we showed that the space $B(X)$ of all bounded real-valued functions on X is a complete metric space when endowed with the uniform norm. We also showed that the space of bounded continuous functions $BC(X)$ is a closed subset of $B(X)$ and is hence also complete. We noted above that $C_0(X) \subset BC(X)$.

To show $C_0(X)$ is complete, let $\{f_n \in C_0(X)\}$ be a uniformly-Cauchy sequence. By completeness of $BC(X)$, there is an $f \in BC(X)$ so that $f_n \to f$ uniformly. It remains to show that $f \in C_0(X)$. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there is an N so that $n > N$ implies that $\|f_n - f\| < \frac{\epsilon}{2}$. Fix some $n > N$. Because $f_n \in C_0(X)$, there is a compact $K \subset X$ so that $|f_n(x)| < \frac{\epsilon}{2}$ when $x \notin K$. Then for $x \notin K$, by the triangle inequality, we have

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This demonstrates that $f \in C_0(X)$.