Math 70100: Functions of a Real Variable I
Homework 4, due Wednesday, October 1.

Name: Insert your name here.

1. *(Lang Chapter 2, problem 20)* Recall that a space is called *second countable* if it has a countable base for its topology. (Lang calls this notion separable.) A topological space is *metrizable* if it has a metric which induces the same topology on the space. A space is *normal* if it is Hausdorff and for any two disjoint closed sets A and B there are open sets U and V with $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

   Prove that a normal separable space X is metrizable. (Follow the hint suggested by Lang.) (This is the *Urysohn Metrization Theorem.*)

2. *(Royden-Fitzpatrick §12.1 # 6)* Let X be a set and $\mathcal{T}$ be a topology on X. Let $C(X)$ denote the collection of all continuous real-valued functions on $(X, \mathcal{T})$, and let $\mathcal{W}$ denote the weak topology induced by $C(X)$. (That is $\mathcal{W}$ is the coarsest topology on X so that every $f \in C(X)$ is continuous.) Show that if $(X, \mathcal{T})$ is normal, then the two topologies are identical.

3. *(Following Rudin's Real and Complex Analysis, pp. 69)* Let X be a locally compact Hausdorff space. (Recall this means that every $x \in X$ has a compact neighborhood.)

   A *compactly supported function on* X is a function $f : X \to \mathbb{R}$ so that there is a compact set $K \subset X$ so that $f(x) = 0$ for $x \notin K$. We write $C_c(X)$ to denote the collection of all continuous compactly supported functions on X.

   A function $f : X \to \mathbb{R}$ *vanishes at infinity* if for all $\epsilon > 0$ there is a compact set $K \subset X$ so that $|f(x)| < \epsilon$ for $x \notin K$. We write $C_0(X)$ to denote the collection of all continuous functions which vanish at ∞.

   We endow these spaces with the uniform (or sup) norm. Observe that $C_c(X) \subset C_0(X)$.

   (a) Show that $C_c(X)$ is dense in $C_0(X)$. (*Hint:* You need the version of Urysohn's lemma given in class: If X is locally compact and Hausdorff, and $K \subset U \subset X$ with K compact and U open, then there is a continuous $f : X \to [0, 1]$ so that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \notin U$.)

   (b) Show that $C_0(X)$ is a Banach space (i.e., that it is complete).

   Together, this shows that $C_0(X)$ is the metric completion of $C_c(X)$.