Math 70100: Functions of a Real Variable I
Homework 3, due Tuesday, September 23rd.

1. The Hilbert cube is the countable product $H = [0,1]^{\mathbb{N}}$ of all functions $\mathbb{N} \to [0,1]$ endowed with the product topology. Give a direct proof that the Hilbert cube is sequentially compact. That is, given a sequence $\{\alpha^n \in H\}_{n \in \mathbb{N}}$, find a convergent subsequence. (*Hint:* You may want to use a version of the Cantor diagonal argument.)

Solution: Let $\{\alpha^n \in H\}_{n\in\mathbb{N}}$ be any sequence in H. We will show it has a convergent sequence. Write $\alpha^n = \{a^n_k\}_{k\in\mathbb{N}}$.

We will inductively define a subsequences of $\{\alpha^n \in H\}_{n\in\mathbb{N}}$. Let $n_0(m) = m$. Then $\alpha^m = \alpha^{n_0(m)}$. This defines a first subsequence of $\{\alpha_n\}$ namely

$$\left\{\alpha^{n_0(m)}\right\}_{m\in\mathbb{N}},$$

though we are just using new notation for the same sequence. Now let us suppose that $\alpha^{n_{i-1}(m)}$ is defined for some $i \in \mathbb{N}$. Observe that the i-th term in the sequence $\alpha^{n_{i-1}(m)}$ is $a_i^{n_{i-1}(m)}$. The sequence $\{a_i^{n_{i-1}(m)}\}_{m\in\mathbb{N}}$ is a sequence in $[0,1]$. By the compactness of $[0,1]$, there is a convergent subsequence

$$\{a_i^{n_i(m)}\}_{m\in\mathbb{N}} \quad \text{of} \quad \{a_i^{n_{i-1}(m)}\}_{m\in\mathbb{N}}$$

, i.e., $\{n_i(m) \,:\, m \in \mathbb{N}\}$ is an infinite subset of $\{n_{i-1}(m) \,:\, m \in \mathbb{N}\}$, $n_{i-1}(m)$ is an increasing function $\mathbb{N} \to \mathbb{N}$, and $\lim_{m\to\infty} a_i^{n_i(m)}$ is some $b_i \in [0,1]$.

Remarks: We make a couple of remarks to clarify the argument if you haven't studied a diagonal argument before. We have determined an integer $n_i(m)$ for each $i \geq 0$ and $m \in \mathbb{N}$. Because each n_i indexes a subsequence, the quantity $n_i(m)$ strictly increases as we fix i and increase m, i.e., $n_i(m) > n_i(m+1)$ for all i and m. As we increment i we get a subsequence, so $n_{i+1}(m) \geq n_i(m)$ for all i and m. In particular, note that the diagonal is strictly increasing: $n_{k+1}(k+1) > n_k(k)$ for all k. The sequence indexed by $n_k(k)$ is the diagonal subsequence of $\{\alpha^n\}$.

By induction, we have define a subsequence for each integer $i \geq 0$, $\{\alpha^{n_i(m)}\}_{m\in\mathbb{N}}\}$. Consider the diagonal subsequence given by

$$\{\alpha^{n_k(k)}\}_{k\in\mathbb{N}}.$$

We claim that this sequence converges in H. To do this, it is sufficient to show that for each $i \in \mathbb{N}$ that

$$\lim_{k\to\infty} a_i^{n_k(k)} = b_i,$$

where b_i was the limit of $\{a_i^{n_i(m)}\}_{i\in\mathbb{N}}$ as described above. (We explain this in the remark below.) Note that limiting behavior does not change if we drop finitely many elements of the sequence. Because for each $k \geq i$, the sequence $\{\alpha^{n_k(m)}\}_{m\in\mathbb{N}}\}$ is a subsequence of $\{\alpha^{n_i(m)}\}_{m\in\mathbb{N}}\}$, for each $k \geq i$, there is an m so that $n_k(k) = n_i(m)$. In particular, the portion of the diagonal subsequence,

$$\{\alpha^{n_k(k)}\}_{k\geq i} \text{ is a subsequence of } \{\alpha^{n_i(m)}\}_{m\in\mathbb{N}}\}.$$

Then because subsequences of convergent sequences have the same limits, we know

$$\lim_{k\to\infty} a_i^{n_k(k)} = b_i.$$

Remark: We have shown that $\{a_i^{n_k(k)}\}_{k\in\mathbb{N}}$ converges to b_i for every $i \in \mathbb{N}$, but we haven't explained that $\{\alpha^{n_k(k)}\}_{k\in\mathbb{N}}$ converges to $\beta = \{b_i\}_{i\in\mathbb{N}}$. Let $N \subset H$ be a neighborhood of β. Note we are using the product topology, which is the coarsest topology so that the projection functions $\pi_i : H \to [0,1]$ are continuous. The pre-images of open sets under these projections form a subbasis, so there is a collection of indices $\{i(j) : j \in J\}$ with J a finite set and a choice of $U_j \subset [0,1]$ so that

$$\beta \in \bigcap_{j\in J} \pi_{i(j)}^{-1}(U_j) \subset N.$$

Each U_j is an open set containing $b_{i(j)}$, so for each $j \in J$ there is an integer K_j so that

$$a_{i(j)}^{n_k(k)} \in U_j \quad \text{for } k > K_j.$$

So setting $K = \max\{K_j : j \in J\}$, we see that

$$\alpha^{n_k(k)} \in \bigcap_{j\in J} \pi_{i(j)}^{-1}(U_j) \subset N \quad \text{for } k > K.$$

This verifies by definition that $\{\alpha^{n_k(k)}\}_{k\in\mathbb{N}}$ converges to β.

2. *(Modified from Pugh Chapter 2 #79)* A space X is locally path-connected if given any $x \in X$ and any open set $U \subset X$ containing x, there is an open set $V \subset U$ containing x which is path-connected.

   Let X be a topological space which is non-empty, compact, locally path-connected and connected. Prove that X is path-connected.

Solution: Let X be as stated. Let $x \in X$. By applying the definition of local path-connectivity to $x \in X$, which is contained in the open set X, we see that there is an path-connected open set V_x containing $x \in X$. Then $\{V_x : x \in X\}$ is an open cover of X, so there is a finite subcover $\{V_1 = V_{x_1}, \ldots, V_n = V_{x_n}\}$.

We claim that for every $j,k \in \{1,\ldots,n\}$, there is an list with $i(0) = j$, $i(m) = n$ with $V_{i(a)} \cap V_{i(a+1)} = \emptyset$ for $0 \le a < m$.

We define an equivalence relation on $\{1,\ldots,n\}$ by $j \sim k$ defined if there is a list

$$i(0), i(1), \ldots, i(m) \in \{1,\ldots,n\}$$

with $i(0) = j$, $i(m) = k$, and $V_{i(a)} \cap V_{i(a+1)} = \emptyset$ for $0 \le a < m$.

We claim that $j \sim k$ for each pair $j, k \in \{1, \ldots, n\}$. Suppose to the contrary, $j \not\sim k$ for some j and k. Let $[j]$ and $[k]$ denote their equivalence classes. Then,

$$A = \bigcup_{i \in [j]} V_i \quad \text{and} \quad B = \bigcup_{i \in [k]} V_i$$

are disjoint non-empty open sets. It then follows that X is disconnected. But, since X is connected, this is a contradiction.

Now we will show that X is path-connected. Let $x, y \in X$. Then, there are $j, k \in \{1, \ldots, n\}$ so that $x \in V_j$ and $y \in V_k$. Since $j \sim k$, there is a list

$$i(0), i(1), \ldots, i(m) \in \{1, \ldots, n\}$$

with $i(0) = j$, $i(m) = k$, and $V_{i(a)} \cap V_{i(a+1)} = \emptyset$ for $0 \leq a < m$. Then we can define a sequence of points $x_0, x_1, \ldots, x_{m+1}$ so that $x_0 = x$, $x_{m+1} = y$ and $x_{a+1} \in V_{i(a)} \cap V_{i(a+1)}$ for $1 \leq a < m$. Observe that for each $a \in \{0, \ldots, m\}$ we have $x_a, x_{a+1} \in V_{i(a)}$. So, we can define a path $\gamma : [0, 1] \to X$ with

$$\gamma(\frac{a}{m}) = x_a \quad \text{for } a \in \{0, 1, \ldots, m\}.$$

On each interval $[\frac{a}{m}, \frac{a+1}{m}]$, we define γ to joint x_a to x_{a+1} via a path in $V_{i(a)}$, which is guaranteed to exist by the path-connectivity of $V_{i(a)}$.

3. Let X be a compact metric space and let $\mathcal{U}$ be an open cover of X. Prove that there is an $\epsilon > 0$ so that for every $x \in X$ there is a $U \in \mathcal{U}$ containing the open ball of radius ϵ about x. (Such an $\epsilon > 0$ is called a *Lebesgue number* for the cover.)

Solution: For $x \in X$ and $r > 0$, let $B_r(x)$ denote the open ball of radius r about x. For each $x \in X$, let

$$r(x) = \sup \ \{r > 0 \ : \ \text{there is a } U \in \mathcal{U} \text{ with } B_r(x) \subset U\}.$$

Observe that for each $x \in X$, we have $r(x) > 0$ because by definition of open cover there is at least one $U \in \mathcal{U}$ containing x and this open set must contain an open ball about x.

We also claim that $r(x)$ is continuous. In fact it is Lipschitz, which is a strengthening of uniform continuity. We will show that for each $\delta > 0$, and each $x, y \in X$ we have $d(x, y) < \delta$ implies $|r(x) - r(y)| < 2\delta$. It is sufficent to show that $d(x, y) < \delta$ implies $r(y) > r(x) - 2\delta$, since the two inequalities

$$r(y) > r(x) - 2\delta \quad \text{and} \quad r(x) > r(y) - 2\delta$$

are equivalent to the statement $|r(x) - r(y)| < 2\delta$.

Suppose $d(x, y) < \delta$. We claim $r(y) > r(x) - 2\delta$. We can suppose $r(x) > 2\delta$ otherwise the claim is vacuous. By definition of $r(x)$, there is an open set U containing the open ball B of radius $r(x) - \delta$ about x. Since $r(x) > 2\delta$ and $d(x,y) < \delta$ we see that $y \in B$. Observe that by the triangle inequality, the open ball B' about y of radius $r(x) - \delta - d(x,y)$ is contained in B. So, $B' \subset B \subset U$. We conclude by definition of $r(y)$ that

$$r(y) \geq r(x) - \delta - d(x,y) > r(x) - 2\delta$$

as claimed.

Now we know that $r(x)$ is a continuous function. Therefore it attains its minimum. That is, there is an $x_0 \in X$ so that

$$r(x_0) \leq r(x) \quad \text{for all } x \in X.$$

As noted above $r(x_0)$ is positive. Any positive number smaller than $r(x_0)$ is a Lebesgue number for the cover. (In addition, $r(x_0)$ is the largest possible Lebesgue number for the cover. Though, $r(x_0)$ may not be a Lebesgue number.)

4. If A and B are subsets of $\mathbb{R}$, then we define

$$A + B = \{a + b \ : \ a \in A \text{ and } b \in B\} \subset \mathbb{R}.$$

Let C be the standard middle third Cantor set. Prove that $C + C = [0,2]$. (*Hint*: Consider ternary expansions.)

Solution: Let $t \in [0,2]$. We will find two points $x, y \in C$ so that $x + y = t$. Because x and y lie in the middle third Cantor set, they each have a ternary expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \quad \text{and} \quad y = \sum_{i=1}^{\infty} \frac{y_i}{3^i}$$

with $x_i, y_i \in \{0,2\}$ for all $i \in \mathbb{N}$.

We will determine the sequences $\{x_i \in \{0,2\}\}_{i \in \mathbb{N}}$ and $\{y_i \in \{0,2\}\}_{i \in \mathbb{N}}$ inductively so that for any integer $n \geq 0$, we have defined x_i and y_i for integers i with $1 \leq i \leq n$ so that the following equation is satisfied:

$$3^n\big(t - \sum_{i=1}^{n} \frac{x_i}{3^i} - \sum_{i=1}^{n} \frac{y_i}{3^i}\big) \in [0,2]. \tag{1}$$

This is trivially true when $n = 0$, because in this case the sums are taken to be zero, and we do not claim to have defined any of x_i or y_i.

Now suppose that equation 1 is true for some $n \geq 0$. In particular, we are supposing x_i and y_i have already been defined for each i with $1 \leq i \leq n$. Let q_n be the quantity in equation 1. Then we define x_{n+1} and y_{n+1} according to the following rule:

$$(x_{n+1}, y_{n+1}) = \begin{cases} (0,0) & \text{if } q_n \in [0, \frac{2}{3}). \\ (2,0) & \text{if } q_n \in [\frac{2}{3}, \frac{4}{3}). \\ (2,2) & \text{if } q_n \in [\frac{2}{3}, \frac{4}{3}). \end{cases}$$

The key point with this definition is that

$$q_n - \frac{x_{n+1}}{3} - \frac{y_{n+1}}{3} \in [0, \frac{2}{3}]$$

no matter which case we are in. Now we will verify that equation 1 is satisfied when n is replaced by $n+1$. In this case, the quantity becomes:

$$\begin{aligned} 3^{n+1}\left(t - \sum_{i=1}^{n+1} \tfrac{x_i}{3^i} - \sum_{i=1}^{n+1} \tfrac{y_i}{3^i}\right) &= 3\left(3^n\left(t - \sum_{i=1}^{n} \tfrac{x_i}{3^i} - \sum_{i=1}^{n} \tfrac{y_i}{3^i}\right) - \tfrac{x_{n+1}}{3} - \tfrac{y_{n+1}}{3}\right) \\ &= 3(q_n - \tfrac{x_{n+1}}{3} - \tfrac{y_{n+1}}{3}), \end{aligned}$$

which lies in $[0, 2]$ because $q_n - \frac{x_{n+1}}{3} - \frac{y_{n+1}}{3} \in [0, \frac{2}{3}]$. This verifies the inductive hypothesis.

Now we claim that t equals $x + y$, where x and y are the numbers with ternary expansion $\{x_i\}$ and $\{y_i\}$. This is a consequence of equation 1, which can be written as

$$0 \leq t - \sum_{i=1}^{n} \frac{x_i}{3^i} - \sum_{i=1}^{n} \frac{y_i}{3^i} \leq \frac{2}{3^n} \quad \text{for all } n \geq 0.$$

Then by the "Squeeze theorem,"

$$t - x - y = \lim_{n\to\infty} t - \sum_{i=1}^{n} \frac{x_i}{3^i} - \sum_{i=1}^{n} \frac{y_i}{3^i} = 0.$$