Math 70100: Functions of a Real Variable I
Homework 3, due Tuesday, September 23rd.

1. The Hilbert cube is the countable product $H = [0,1]^{\mathbb{N}}$ of all functions $\mathbb{N} \to [0,1]$ endowed with the product topology. Give a direct proof that the Hilbert cube is sequentially compact. That is, given a sequence $\{\alpha^n \in H\}_{n\in\mathbb{N}}$, find a convergent subsequence. (*Hint:* You may want to use a version of the Cantor diagonalization argument.)

2. *(Modified from Pugh Chapter 2 #79)* A space X is locally path-connected if given any $x \in X$ and any open set $U \subset X$ containing x, there is an open set $V \subset U$ containing x which is path-connected.

   Let X be a topological space which is non-empty, compact, locally path-connected and connected. Prove that X is path-connected.

3. Let X be a compact metric space and let $\mathcal{U}$ be an open cover of X. Prove that there is an $\epsilon > 0$ so that for every $x \in X$ there is a $U \in \mathcal{U}$ containing the open ball of radius ϵ about x. (Such an $\epsilon > 0$ is called a *Lebesgue number* for the cover.)

4. If A and B are subsets of $\mathbb{R}$, then we define

$$A + B = \{a + b \ : \ a \in A \text{ and } b \in B\} \subset \mathbb{R}.$$

   Let C be the standard middle third Cantor set. Prove that $C + C = [0,2]$. (*Hint*: Consider ternary expansions.)