# Math 70100: Functions of a Real Variable I
## Homework 2, due Wednesday, September 17th.

1. *(Modified from Folland 4.1.13)* Suppose $X$ is a topological space and $A \subset X$ is dense. Prove that if $U \subset X$ is open, then $\bar{U} = \overline{U \cap A}$, where $\bar{\cdot}$ denotes closure.

> **Solution:** Let $B \subset X$. Recall $\bar{B} = X \setminus \big(\mathrm{Int}(X \setminus B)\big)$. By definition of the interior, then $x \notin \bar{B}$ if and only if there is an open set $V \subset X$ with $x \in V$ and $V \cap B = \emptyset$. Thus, to see that $\bar{U} = \overline{U \cap A}$, it suffices to check that for any open set $V \subset X$, we have $V \cap U = \emptyset$ if and only if $V \cap U \cap A = \emptyset$. One way is clear: if $V \cap U = \emptyset$, then clearly $V \cap U \cap A = \emptyset$. Now suppose that $V \cap U \neq \emptyset$. Then $V \cap U$ is open and non-empty, so by density of $A$, there is an $a \in V \cap U \cap A$. In particular, $V \cap U \cap A$ is non-empty as required.

2. (From Zakeri's Homework 2) Give a direct proof that the interval $[0, 1]$ is compact. (*Hint:* Let $\mathcal{U}$ be an open cover. Define

$$S = \{x \in [0, 1] \ : \ [0, x] \text{ is covered by finitely many } U \in \mathcal{U}\}.)$$

Prove that $S = [0, 1]$.)

> **Solution:** Let $t = \sup\ S$. We claim that $t \in S$. Since $t \in [0, 1]$, there is an open set $U_* \in \mathcal{U}$ so that $t \in U_*$. Because $U_*$ is open and contains $t$, there is an $\epsilon > 0$ so that $s \in U_*$ whenever $t - \epsilon < s < t$. Then because $t = \sup\ S$, there is an $s \in S$ with $t - \epsilon < s < t$. Then $[0, s]$ has a finite covering by some $U_1, \ldots, U_n \in \mathcal{U}$. Now observe that $\{U_*\} \cup \{U_i \ : \ 1 \le i \le n\}$ is a finite covering of $[0, t]$, so $t \in S$ as claimed.
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> Again let $t = \sup\ S$. Now we claim that $t = 1$. Suppose to the contrary that $t < 1$. Then, $[0, t]$ has a finite covering by some $V_1, \ldots, V_m \in \mathcal{U}$ with $[0, t] \subset \bigcup_{i=1}^m V_i$. But then all of $\bigcup_{i=1}^m V_i$ is covered by this finite collection, and $\bigcup_{i=1}^m V_i$ contains $t$ and since it is open also contains real numbers bigger than $t$. But this contradicts the definition of $t$ as $\sup\ S$. We conclude that $t = 1$. From the prior paragraph, we know that $1 \in S$, so by definition of $S$, the interval $[0, 1]$ is covered by finitely many elements of $\mathcal{U}$.

3. *(Modified from Lang II.5.1a)* Let $X$ and $Y$ be compact Hausdorff topological spaces. Prove that $f : X \to Y$ is continuous if and only if its graph is closed in $X \times Y$. (The *graph* of $f$ is the set

$$\Gamma = \{(x, y) \in X \times Y \ : \ y = f(x)\}.)$$

(*Remark:* More generally the result is true if $X$ is just a topological space and $Y$ is a compact Hausdorff space. This is the *closed graph theorem.*)

**Solution:** *Remark 1*: In solving the problem, it may be useful to note that the statement fails in the absence of compactness. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1/x$ if $x < 0$ and $f(x) = 0$ if $x \geq 0$. This function has a closed graph but is not continuous.

Let $X$ and $Y$ be compact Hausdorff spaces. Let $f : X \to Y$ be a function and let $\Gamma$ be its graph.

First suppose that $f$ is continuous. We will show its graph $\Gamma$ is closed by showing its compliment is open. To show $(X \times Y) \smallsetminus \Gamma$ is open, it is sufficient to show that for any point $(x, y) \in (X \times Y) \smallsetminus \Gamma$, there is an open set $U \subset X \times Y$ so that $(x, y) \in U$ and $U \cap \Gamma = \emptyset$. Choose $(x, y) \in (X \times Y) \smallsetminus \Gamma$. Then $y \neq f(x)$. Then since $Y$ is Hausdorff, there are disjoint open sets $V_1$ and $V_2$ so that $f(x) \in V_1$ and $y \in V_2$. Continuity of $f$ implies that $f^{-1}(V_1) \subset X$ is an open neighborhood of $x$. Then for any point $x' \in f^{-1}(V_1)$, we have $f(x) \in V_1$. In particular the no point in $f^{-1}(V_1)$ has an image in $V_2$. It follows that $f^{-1}(V_1) \times V_2$ is disjoint from the graph $\Gamma$. Further, the product of open sets is open. This verifies that the compliment of $\Gamma$ is open and therefore that $\Gamma$ is closed.

Now suppose that the graph $\Gamma$ is closed. Take $V \subset Y$ open. We will show that $f^{-1}(V)$ is open. Fix some $x \in f^{-1}(V)$ so that $f(x) \in V$. It is sufficient to find a neighborhood $A$ of $x$ so that $f(a) \subset V$ for $a \in A$. Since $\Gamma$ is closed, for each point $(p, q) \in (X \times Y) \smallsetminus \Gamma$, there are open sets $A_{p,q} \subset X$ and $B_{p,q} \subset Y$ with $p \in A_{p,q}$, $q \in B_{p,q}$ and $(A_{p,q} \times B_{p,q}) \cap \Gamma = \emptyset$. (This uses the fact that sets of this type form a basis for the product topology.) In particular, make a choice of such sets $A_{x,y}$ and $B_{x,y}$ for each pair of points $(x, y)$ where $x \in f^{-1}(V)$ is fixed as above and $y \notin V$. Observe that $Y \smallsetminus V$ is a closed subset of a compact space and therefore compact. The sets $\{B_{x,y} \ : \ y \notin V\}$ form an open cover of $Y \smallsetminus V$, so there is a finite subcover of the form

$$\{B_{x,y_1}, \ldots, B_{x,y_n}\}$$

where $y_1, \ldots, y_n$ is a list of elements of $Y \smallsetminus V$. Now set

$$A = A_{x,y_1} \cap \ldots \cap A_{x,y_n}.$$

Observe that $x \in A$ because $x \in A_{x,y}$ for all $y \notin V$. We claim that if $a \in A$ then $f(a) \in V$, which verifies the continuity of $f$ because $A$ is open and contains $x$. Suppose to the contrary that $f(a) \notin V$. Then because we have a finite cover of $Y \smallsetminus V$, $f(a) \in B_{x,y_i}$ for some $i \in \{1, \ldots, n\}$. Then $a \in A_{x,y_i}$ and $f(a) \in B_{x,y_i}$, but $A_{x,y_i} \times B_{x,y_i}$ was defined to be disjoint from $\Gamma$. This contradicts the observation that $\big(a, f(a)\big)$ lies in the graph.

4. *(Modified from Lang II.5.1b)* A function $f : X \to Y$ between metric spaces is *uniformly continuous* if for all $\epsilon > 0$, there is a $\delta > 0$ so that $d_X(x_1, x_2) < \delta$ implies $d_Y\big(f(x_1), f(x_2)\big) < \epsilon$ for all $x_1, x_2 \in X$.

   Let $Y$ be a complete metric space and $X$ be a metric space. Let $A \subset X$. Let $f : A \to Y$ be uniformly continuous, and let $\bar{A} \subset X$ denote the closure of $A$. Show that there exists a unique extension of $f$ to a continuous map $\bar{f} : \bar{A} \to Y$, and show that $\bar{f}$ is uniformly continuous. (You may assume that $X$ and $Y$ are subsets of Banach spaces if you wish, in order to write the distance function in terms of the absolute value sign.)

**Solution:** *Remark:* The main idea here is that uniformly continuous functions send Cauchy sequences to Cauchy sequences. Also note that all convergent sequences are Cauchy.

Suppose as above that $X$ and $Y$ are metric spaces with $Y$ complete. Let $A \subset X$ and let $f : A \to Y$ be uniformly continuous.

We first need to define our extension $\bar{f}$ at each point $a \in \bar{A} \setminus A$. For each such $a$, we choose an a sequence of points $\{x_n \in A\}$ which converge to $a$. If we want $\bar{f}$ to be continuous at $a$, then the sequence $\{f(x_n)\}$ must converge and $\bar{f}(a)$ must be the limit. (Otherwise we get a contradiction to the definition of continuity of $\bar{f}$ at $a$.) This verifies uniqueness of $\bar{f}$, since the values of $\bar{f}$ at each point in $\bar{A} \setminus A$ are uniquely determined. Now we will check that the sequence $\{f(x_n)\}$ converges. Because $Y$ is complete, it is sufficient to check that $\{f(x_n)\}$ is Cauchy. Observe that because $\{x_n\}$ converges to $a$, this sequence is Cauchy. Now we will verify that $\{f(x_n)\}$ is Cauchy. Choose $\epsilon > 0$. Then, because $f$ is uniformly continuous, there is a $\delta > 0$ so that $d_X(x, x') < \delta$ implies $d_Y\big(f(x), f(x')\big) < \epsilon$. Then because $\{x_n\}$ is Cauchy, there is an $N$ so that $m, n > N$ implies $d_X(x_n, x_m) < \delta$. Taken together we see that $m, n > N$ implies $d_Y\big(f(x_m), f(x_n)\big) < \epsilon$, as required.

It remains to check that $\bar{f}$ is uniformly continuous. Choose $\epsilon > 0$. Then because $f$ is uniformly continuous, there is a $\delta > 0$ so that for each $a, b \in A$ with $d_X(a, b) < \delta$, we have $d_Y\big(f(a), f(b)\big) < \frac{\epsilon}{3}$. Now let $c, d \in \bar{A}$ with $d_X(c, d) < \frac{\delta}{3}$. We claim that $d_Y\big(\bar{f}(c), \bar{f}(d)\big) < \epsilon$, which will verify the definition of uniform continuity. Suppose $c \in \bar{A} \setminus A$. Above we chose a sequence $\{x_n \in A\}$ converging to $c$, and we know $\{f(x_n)\}$ converges to $\bar{f}(c)$. So, there is an $N$ so that $n > N$ implies $d_X(c, x_n) < \frac{\delta}{3}$ and $d_Y\big(\bar{f}(c), f(x_n)\big) < \frac{\epsilon}{3}$. Set $a = x_n \in A$ for some $n > N$. Then

$$d_X(c, a) < \frac{\delta}{3} \quad \text{and} \quad d_Y\big(\bar{f}(c), f(a)\big) < \frac{\epsilon}{3}.$$

Also suppose that $d \in \bar{A} \setminus A$. In a similar manner, we find $b \in A$ with

$$d_X(d, b) < \frac{\delta}{3} \quad \text{and} \quad d_Y\big(\bar{f}(d), f(b)\big) < \frac{\epsilon}{3}.$$

Now observe that by the triangle inequality,

$$d_X(a, b) \le d_X(a, c) + d_X(c, d) + d_X(d, b) = \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

So, by our use of uniform continuity of $f$, we see that $d_Y\big(f(a), f(b)\big) < \frac{\epsilon}{3}$. Then,

$$d_Y\big(\bar{f}(c), \bar{f}(d)\big) \le d_Y\big(\bar{f}(c), f(a)\big) + d_Y\big(f(a), f(b)\big) + d_Y\big(f(b), \bar{f}(d)\big) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This verifies that $d_X(c, d) < \frac{\delta}{3}$ implies $d_Y\big(\bar{f}(c), \bar{f}(d)\big) < \epsilon$ in the case when $c, d \in \bar{A} \setminus A$. Simpler arguments can be used when $c \in A$ or $d \in A$. In the case when $c \in A$ for instance, the above inequalities all hold with $a$ taken to equal $c$.

5. *(Lang II.5.12)* Let $U$ be an open subset of a normed vector space. Show that $U$ is connected if and only if $U$ is path (or arcwise) connected. (Recall that if a topological space is path connected, then it is connected. See Proposition 2.7. You do not need to prove this.) (*Hint*: define

the notion of a path-component, which is analogous to the notion of connected component.)

**Solution:** Suppose that $U$ is not path connected. We will show that $U$ is not connected.

We define a relation on $U$. Let $p, q \in U$, and say they are *joined by a path* if there is a path $\gamma : [0,1] \to U$ with $\gamma(0) = p$ and $\gamma(1) = q$. This is an equivalence relation: It is reflexive because the constant function $\gamma(t) = p$ is a path. It is symmetric because if $\gamma(t)$ is a path, then so is $\gamma(1-t)$. It is transitive because if $\gamma$ joins $p$ to $q$ and $\eta$ joins $q$ to $r$, then the path

$$t \mapsto \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \eta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

joins $p$ to $r$.

Because this is an equivalence relation, it partitions $U$ into equivalence classes. (The equivalence class of $p \in U$ is called the *path-component of* $p$.) For $p \in P$, let $[p]$ denote the path-component of $p$. There are at least two equivalence classes because $U$ is not path connected. Endow the collection $\{[p] \; : \; p \in U\}$ with the discrete topology. We claim that the map $p \mapsto [p]$ is continuous. This implies that $U$ is disconnected, since the image contains at least two distinct points and the continuous image of a connected set in a discrete space can only consist of one point. See Proposition 2.2.

To see that $p \mapsto [p]$ is continuous, take an arbitrary point $q \in U$ and consider its path-component $[q]$. We claim that $[q]$ it contains an open neighborhood about $q$. This is equivalent to saying that $p \mapsto [p]$ is continuous at $q$, and because we took $q$ to be arbitrary this implies that $p \mapsto [p]$ is open. Since $q \in U$, $U$ is open, and open balls form a basis for the topology, there is an $\epsilon > 0$ so that the open ball centered at $q$, $B_\epsilon(q)$, is a subset of $U$. We claim that the whole ball $B_\epsilon(q)$ is in the same path component as $q$. Choose $r \in B_\epsilon(q)$ distinct from $q$. Then we can define

$$\gamma(t) = (1-t)q + tr.$$

Clearly $\gamma$ joins $q$ to $r$ and $\gamma(0,1) \subset B_\epsilon(q) \subset U$. It follows that $[r] = [q]$. Because $r \in B_\epsilon(q)$ was arbitrary, we conclude that $[r] = [q]$ for each $r \in B_\epsilon(q)$. This verifies that the preimage of $[q]$ contains $B_\epsilon(q)$ as required to verify the continuity of $p \mapsto [p]$ at $q$.

*Remark:* To be pedantic, we should have shown that $\gamma(t)$ is continuous. Choose $t_0 \in [0,1]$ and $\epsilon > 0$. Observe that

$$\gamma(t) - \gamma(t_0) = (t - t_0)(r - q).$$

So, setting $\delta = \frac{\epsilon}{|r-q|}$, when $|t - t_0| < \delta$, we have

$$|\gamma(t) - \gamma(t_0)| = |(t-t_0)(r-q)| = |t - t_0| \cdot |r - q| < \delta \cdot |r - q| = \epsilon.$$

This verifies the metric definition of continuity.

6. The closed topologist's sine curve is

$$T = \big\{(x, \sin\frac{\pi}{x}) \; : \; 0 < x \le 1\big\} \cup \{(0,y) \; : \; y \in [-1,1]\}.$$

Show that $T$ is connected but not path connected.

**Solution:** We begin by showing $T$ is connected. If $T$ were not connected, then it could be split (non-trivially) into connected components. Observe that the subsets

$$A = \{(0,y) \ : \ y \in [-1,1]\} \quad \text{and} \quad B = \Big\{(x, \sin\frac{\pi}{x}) \ : \ 0 < x \leq 1\Big\}.$$

are both connected since they are continuous images of intervals. It follows that the only possible connected components of $T$ which are proper subsets of $T$ are $A$ and $B$. So, if $T$ were disconnected, it would have to be that $A$ and $B$ are disjoint open subsets. We will show that $A$ is not in fact open in $T$. Recall that in the subspace topology on $T$ that open sets are intersections of open sets in $\mathbb{R}^2$ with $T$. Therefore, to prove that $A$ is not open in $T$, it suffices to show that there is no open set $U \subset \mathbb{R}^2$ so that $U \cap T = A$. Let $U \subset \mathbb{R}^2$ be any open set containing $A$. Then in particular, $U$ contains the point $(0,0)$. Then because $U$ is open, there is an $\epsilon > 0$ so that $(x,0) \in U$ whenever $x < \epsilon$. Then we can choose a integer $n \geq 1$ so that $\frac{1}{n} < \epsilon$. Then

$$(\frac{1}{n}, \sin\frac{\pi}{1/n}) = (\frac{1}{n}, \sin n\pi) = (\frac{1}{n}, 0) \in U \cap B.$$

We conclude that any open set containing $A$ also contains points in $B$, so $A$ is not open. We conclude that $T$ is connected, because there is no way to non-trivially split $T$ into connected components.

Now we will show $T$ is not path connected. If it was path connected, then there would be a continuous $\gamma : [0,1] \to T$ with $\gamma(0) = (1,0)$ and $\gamma(1) = (0,1)$. We now suppose such a $\gamma$ exists, and we will draw a contradiction. Consider the continuous projection $\pi_x : (x,y) \mapsto x$. Since $\gamma$ is continuous, the composition $\pi_x \circ \gamma$ is continuous. We conclude that $J = (\pi_x \circ \gamma)^{-1}(\{0\})$ is closed. Note also that it contains 1, since $\gamma(1) = (0,1)$. Set $t_0 = \inf\ J$. Since $J$ is closed and non-empty, $t_0$ is well defined and lies in $J$.

Observe that for $0 \leq t < t_0$, $\pi_x \circ \gamma(t) > 0$ while $\pi_x \circ \gamma(t_0) = 0$. Since $\pi_x \circ \gamma$ is continuous and $[0, t_0]$ is connected with $\pi_x \circ \gamma(0) = 1$ and $\pi_x \circ \gamma(t_0) = 0$, it must be that $\pi_x \circ \gamma([0,t_0]) = [0,1]$. So in particular, there is a sequence $\{s_n \in [0,t_0]\}$ with $\pi_x \circ \gamma(s_n) = \frac{1}{n}$. Observe that by definition of $T$, we have $\gamma(s_n) = (\frac{1}{n}, 0)$. Then by (sequential) compactness of $\gamma([0,t_0])$, there is an $s \in [0,t_0]$ so that

$$\gamma(s) = \lim_{n\to\infty} \gamma(s_n) = \lim_{n\to\infty} (\frac{1}{n}, 0) = (0,0).$$

Recall in the first sentence of the paragraph we stated that $t_0$ was the only point in $[0,t_0]$ where $\pi_x \circ \gamma(t) = 0$. We conclude that $s = t_0$ and $\gamma(t_0) = (0,0)$. We now make a similar argument for a different sequence. Observe that there is a sequence $\{r_n \in [0,t_0]\}$ with $\pi_x \circ \gamma(s_n) = \frac{2}{4n+1}$. Again by compactness, there is an $r \in [0,t_0]$ so that

$$\gamma(r) = \lim_{n\to\infty} \gamma(r_n) = \lim_{n\to\infty} (\frac{2}{4n+1}, \sin\frac{(4n+1)\pi}{2}) = \lim_{n\to\infty} (\frac{2}{4n+1}, 1) = (0,1).$$

Because $\pi_x \circ \gamma(r) = 0$, we again conclude that $r = t_0$ and therefore $\gamma(t_0) = (0,1)$. We have shown $\gamma(t_0) = (0,0)$ and $\gamma(t_0) = (0,1)$, which is a contradiction.