Math 70100: Functions of a Real Variable I
Homework 2, due Wednesday, September 17th.

1. *(Modified from Folland 4.1.13)* Suppose X is a topological space and $A \subset X$ is dense. Prove that if $U \subset X$ is open, then $\bar{U} = \overline{U \cap A}$, where $\bar{\cdot}$ denotes closure.

2. (From Zakeri's Homework 2) Give a direct proof that the interval $[0,1]$ is compact. (*Hint:* Let $\mathcal{U}$ be an open cover. Define

$$S = \{x \in [0,1] \ : \ [0,x] \text{ is covered by finitely many } U \in \mathcal{U}\}.)$$

   Prove that $S = [0,1]$.)

3. *(Modified from Lang II.5.1a)* Let X and Y be compact Hausdorff topological spaces. Prove that $f : X \to Y$ is continuous if and only if its graph is closed in $X \times Y$. (The *graph* of f is the set

$$\Gamma = \{(x,y) \in X \times Y \ : \ y = f(x)\}.)$$

   (*Remark:* More generally, the result is true if X is just a topological space and Y is a compact Hausdorff space. This is the *closed graph theorem.*)

4. *(Modified from Lang II.5.1b)* A function $f : X \to Y$ between metric spaces is *uniformly continuous* if for all $\epsilon > 0$, there is a $\delta > 0$ so that $d_X(x_1, x_2) < \delta$ implies $d_Y\big(f(x_1), f(x_2)\big) < \epsilon$ for all $x_1, x_2 \in X$.

   Let Y be a complete metric space and X be a metric space. Let $A \subset X$. Let $f : A \to Y$ be uniformly continuous, and let $\bar{A} \subset X$ denote the closure of A. Show that there exists a unique extension of f to a continuous map $\bar{f} : \bar{A} \to Y$, and show that $\bar{f}$ is uniformly continuous. (You may assume that X and Y are subsets of Banach spaces if you wish, in order to write the distance function in terms of the absolute value sign.)

5. *(Lang II.5.12)* Let U be an open subset of a normed vector space. Show that U is connected if and only if U is path (or arcwise) connected. (Recall that if a topological space is path connected, then it is connected. See Proposition 2.7. You do not need to prove this.) (*Hint*: define the notion of a path-component, which is analogous to the notion of connected component.)

6. The closed topologist's sine curve is

$$T = \big\{(x, \sin\frac{\pi}{x}) \ : \ 0 < x \le 1\big\} \cup \{(0,y) \ : \ y \in [-1,1]\}.$$

   Show that T is connected but not path connected.