Math 70100: Functions of a Real Variable I
Homework 1, due Wednesday, September 10th.

1. *(Based on Pugh 2.92)* Let X be a topological space. Recall that a *neighborhood* N of $x \in X$ is a subset $N \subset X$ so that there is an open set $U \subset X$ with $x \in U$ and $U \subset N$. A *boundary point* of a set $A \subset X$ is a point $x \in X$ so that every neighborhood of x intersects both A and $X \setminus A$. The boundary of A, ∂A, is the set of all boundary points of A.

   (a) Show that $\partial A = X \setminus \big(\mathrm{Int}(A) \cup \mathrm{Int}(X \setminus A)\big)$.

   Solution: Recall that $\mathrm{Int}(A)$ is the union of all open sets contained in A. Suppose that $x \notin \mathrm{Int}(A) \cup \mathrm{Int}(X \setminus A)$. This is equivalent to saying that no open set containing x is entirely contained in A or $X \setminus A$. In other words that x is a boundary point of A. This shows $X \setminus \big(\mathrm{Int}(A) \cup \mathrm{Int}(X \setminus A)\big) = \partial A$.

   (b) Explain why ∂A is closed.

   Solution: Interiors of sets are always open, so $\mathrm{Int}(A) \cup \mathrm{Int}(X \setminus A)$ is open and its compliment, ∂A, is closed.

   (c) Show that $\partial\partial A \subset \partial A$.

   Solution: It is more generally true that if $B \subset X$ is a closed set, then $\partial B \subset B$. Because B is closed, its compliment is open. So, $\partial B = B \setminus \mathrm{Int}(B)$, which is clearly inside of B.

   (d) Show that $\partial\partial\partial A = \partial\partial A$.

   Solution: Again, it is more generally true that if $B \subset X$ is a closed set, then $\partial\partial B = \partial B$. Because B is closed, $\partial B = B \setminus \mathrm{Int}(B)$. Since ∂B is closed, we can apply this trick again:

   $$\partial\partial B = \partial\big(B \setminus \mathrm{Int}(B)\big) = \big(B \setminus \mathrm{Int}(B)\big) \setminus \mathrm{Int}\big(B \setminus \mathrm{Int}(B)\big).$$

   But, the interior of $B \setminus \mathrm{Int}(B)$ is contained in the interior of B. So,

   $$\partial\partial B \supset \big(B \setminus \mathrm{Int}(B)\big) \setminus \mathrm{Int}(B) = B \setminus \mathrm{Int}(B) = \partial B.$$

   Using part (c), we have $\partial\partial B \subset \partial B$, so together we see $\partial\partial B = \partial B$.

   (e) Given an example which illustrates that $\partial\partial A$ may not equal ∂A.

   Solution: Let $X = \mathbb{R}$ and $A = \mathbb{Q} \cap [0,1]$. Then $\partial A = [0,1]$ and $\partial\partial A = \{0,1\}$.

2. *(Pugh 2.37)* Let C denote the vector space of continuous functions from $[0,1]$ to $\mathbb{R}$. This space can be endowed with the sup (or L^∞) norm,

   $$|f| = \sup\ \{|f(x)|\ :\ x \in [0,1]\}$$

   or the L^1 norm,

   $$\|f\| = \int_0^1 |f(x)|\, dx.$$

   Consider the identity map between id from $(C, |\cdot|)$ to $(C, \|\cdot\|)$.

(a) Show that id is a continuous. (Thus it is a continuous linear bijection.)

Solution: Solution 1: (Based on the metric definition of continuity.) We will show that id is continuous at each point of C. Let $f \in C$. We will show that for each $\epsilon > 0$, there is a $\delta > 0$ so that $|f - g| < \delta$ implies $\|f - g\| < \epsilon$. Choosing $\delta = \epsilon$ suffices. Suppose $|f - g| < \epsilon$. Let $c = |f - g| \geq 0$. Then, $|f(x) - g(x)| \leq c < \epsilon$ for each $x \in [0, 1]$. So,

$$\|f - g\| = \int_0^1 |f(x) - g(x)|\, dx \leq \int_0^1 c\, dx = c < \epsilon.$$

Solution 2: (Bounded operator argument.) Because we are working with normed vector spaces, it is sufficient to prove that id is a bounded linear operator. So, we will show that there is an $M > 0$ so that for all $f \in C$, we have $\|f\| < M|f|$. Observe that by definition, $|f(x)| \leq |f|$ for all $x \in X$. Therefore,

$$\|f\| = \int_0^1 |f(x)|\, dx \leq \int_0^1 |f|\, dx = |f|.$$

Thus, id is a bounded linear operator (with $M > 1$) and therefore id is continuous.

(b) Show that the inverse id^{-1} is not continuous.

Solution: Recall that for a metric space the sequence definition for continuity is equivalent to the topological definition. We will use the sequence definition here. We will find a sequence of functions $\{f_n\}$ so that $\|f_n\|$ tends to zero (and thus f_n tends to the zero function), but $|f_n|$ does not tend to zero.

Define $f_n(x) = \frac{1}{nx+1}$ for integers $n \geq 1$. Note that f_n is positive and decreasing on $[0, 1]$, so $|f_n| = f_n(0) = 1$ for all n. On the other hand,

$$\|f_n\| = \int_0^1 \frac{1}{nx + 1}\, dx = \left[\frac{1}{n} \ln(nx + 1)\right]_0^1 = \frac{\ln(n + 1)}{n}.$$

One can use L'Hôpital's rule to show that $\|f_n\| \to 0$ as $n \to \infty$.

3. *(Modified from Lang II.5.3a)* Let ℓ^1 be the set of all sequences $\alpha = \{a_n\}_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n\in\mathbb{N}} |a_n|$ converges. Define

$$|\alpha| = \sum_{n\in\mathbb{N}} |a_n|.$$

(a) Prove that $|\cdot|$ is a norm on ℓ^1.

Solution: We will verify that $|\cdot|$ satisfies the definition of a normed vector space. Recall that

$$|\alpha| = \sum_{n\in\mathbb{N}} |a_n| = \lim_{N\to\infty} \sum_{n=1}^{N} |a_n|.$$

First we claim that $|\alpha| \geq 0$, with equality only if $a_n = 0$ for all $n \in \mathbb{N}$. Observe that the sequence of partial sums $\sum_{n=1}^{N} |a_n|$ are all non-negative since they are sums of non-negative numbers. Any limit of non-negative numbers is non-negative, so $|\alpha|$ is non-negative. Now suppose that for some $m \in \mathbb{N}$, we have $a_m \neq 0$. In this case the partial sum $\sum_{n=1}^{m} |a_m|$ is positive. Furthermore,

$$|\alpha| = \sum_{n=1}^{m} |a_m| + \sum_{n=m+1}^{\infty} |a_n|,$$

and the later infinite sum is non-negative by the above remarks. So we conclude that

$$|\alpha| \geq \sum_{n=1}^{m} |a_m| > 0.$$

Second, we will show that for $c \in \mathbb{R}$ we have $|c\alpha| = |c||\alpha|$. This is a basic observation about pulling constants out of sums and limits:

$$\begin{aligned} |c\alpha| &= \lim_{N\to\infty} \sum_{n=1}^{N} |ca_n| = \lim_{N\to\infty} \sum_{n=1}^{N} |c||a_n| \\ &= \lim_{N\to\infty} |c| \sum_{n=1}^{N} |a_n| = |c| \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| = |c||\alpha|. \end{aligned}$$

Third, we need to show that if $\alpha = \{a_n\}_{n\in\mathbb{N}}$ and $\beta = \{b_n\}_{n\in\mathbb{N}}$ then $|\alpha + \beta| \leq |\alpha| + |\beta|$. This follows from from properties of sums and limits, and the triangle inequality:

$$\begin{aligned} |\alpha + \beta| &= \lim_{N\to\infty} \sum_{n=1}^{N} |a_n + b_n| \leq \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| + |b_n| \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| + \sum_{n=1}^{N} |b_n| \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| + \lim_{N\to\infty} \sum_{n=1}^{N} |b_n| = |\alpha| + |\beta|. \end{aligned}$$

(b) Recall that a sequence $\{\alpha_n\}$ in a normed vector space is *Cauchy* if given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $|\alpha_m - \alpha_n| < \epsilon$ for $m, n \geq N$. A normed vector space is *complete* if all Cauchy sequences converge. Show that ℓ^1 is complete with the norm $|\cdot|$.

Solution: Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence in ℓ^1. Each α_n is a sequence of real numbers which we denote by $\{a_{n,k}\}_{k\in\mathbb{N}}$.

Now suppose $\{\alpha_n\}_{n\in\mathbb{N}}$ is Cauchy. We first claim that for any $k \in \mathbb{N}$, the sequence $\{a_{n,k}\}_{n\in\mathbb{N}}$ is a Cauchy sequence (in $\mathbb{R}$). To verify this, fix k and let $\epsilon > 0$. Then since $\{\alpha_n\}_{n\in\mathbb{N}}$ is Cauchy, there is an N so that $n, m > N$ implies $|\alpha_n - \alpha_m| < \epsilon$. Finally observe that for each $n, m > N$,

$$|a_{n,k} - a_{m,k}| \leq \sum_{j\in\mathbb{N}} |a_{n,j} - a_{m,j}| = |\alpha_n - \alpha_m| < \epsilon.$$

In particular, $|a_{n,k} - a_{m,k}| < \epsilon$ for $m, n > N$ as needed to show that $\{a_{n,k}\}_{n\in\mathbb{N}}$ is Cauchy.

The previous paragraph showed that for each $k \in \mathbb{N}$, $\{a_{n,k}\}_{n\in\mathbb{N}}$ is Cauchy. Since the real numbers are complete, this sequence has a limit, call it b_k. So we have a sequence $\beta = \{b_k\}_{k\in\mathbb{N}}$ with the property that $\lim_{n\to\infty} a_{n,k} = b_k$ for all k.

It remains to show that $\beta \in \ell^1$ and that $\alpha_n \to \beta$ as $n \to \infty$ in the ℓ^1-norm topology.

We make the following:
Claim 1. If $\epsilon > 0$ and N is such that for $n, m > N$ we have $|\alpha_n - \alpha_m| < \epsilon$, then for $n > N$, $|\alpha_n - \beta| < 3\epsilon$. (Note that the infinite sums of a non-negative sequence such as

$|\alpha_n - \beta| = \sum_{k\in\mathbb{N}} |a_{n,k} - b_k|$ always exists in the sense that it must converge to either a real number or $+\infty$.)

Suppose that Claim 1 is false. Then we can choose $\epsilon > 0$ and N so that for $n, m > N$ we have $|\alpha_n - \alpha_m| < \epsilon$, but there is an $n_0 > N$ so that $|\alpha_{n_0} - \beta| \geq 3\epsilon$. Then since

$$|\alpha_{n_0} - \beta| = \sum_{k\in\mathbb{N}} |a_{n_0,k} - b_k| = \lim_{K\to\infty} \sum_{k=1}^{K} |a_{n_0,k} - b_k| \geq 3\epsilon,$$

there is a K so that

$$\sum_{k=1}^{K} |a_{n_0,k} - b_k| > 2\epsilon.$$

Now observe that for any $k \in \mathbb{N}$ with $1 \leq k \leq K$, we have that $a_{m,k} \to b_k$ as $m \to \infty$. Because this is only a finite list of values of k, we can find an M so that $|a_{m,k} - b_k| < \frac{\epsilon}{K}$ for $m > M$ and each k with $1 \leq k \leq K$. By the triangle inequality, for any $m > M$,

$$\begin{aligned}
\sum_{k=1}^{K} |a_{n_0,k} - b_k| &\leq \sum_{k=1}^{K} \left(|a_{n_0,k} - a_{m,k}| + |a_{m,k} - b_k|\right) \\
&\leq \sum_{k=1}^{K} \left(|a_{n_0,k} - a_{m,k}| + \tfrac{\epsilon}{K}\right) \\
&= \epsilon + \sum_{k=1}^{K} |a_{n_0,k} - a_{m,k}| \leq \epsilon + |\alpha_{n_0} - \alpha_m| < \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

But this contradicts the earlier statement that $\sum_{k=1}^{K} |a_{n_0,k} - b_k| > 2\epsilon$.

We will now show that $\beta \in \ell^1$. We need to show that $\sum_{k\in\mathbb{N}} |b_k| < \infty$. Since $\{\alpha_n\}$ is Cauchy, there is an N so that $n, m > N$ implies $|\alpha_n - \alpha_m| < 1$. Then by Claim 1, we know that for $n > N$, we have $|\alpha_n - \beta| < 3$. Fix such an n. Then by the triangle inequality,

$$\sum_{k\in\mathbb{N}} |b_k| \leq \sum_{k\in\mathbb{N}} \left(|b_k - a_{n,k}| + |a_{n,k}|\right) \leq 3 + |\alpha_n| < \infty.$$

Finally, we need to show that $\alpha_n \to \beta$. Choose $\epsilon > 0$. We will find an N so that $n > N$ implies $|\alpha_n - \beta| < \epsilon$. Since $\{\alpha_n\}$ is Cauchy, we can find an N so that $n, m > N$ implies $|\alpha_n - \alpha_m| < \frac{\epsilon}{3}$. By Claim 1, for this N, we have $n > N$ implies that $|\alpha_n - \beta| < \epsilon$ as desired.

4. *(Lang II.13)* The *diagonal* Δ is the set of all points (x, x).

   (a) Show that a space X is Hausdorff if and only if the diagonal is closed in $X \times X$.

Solution: Because this is a finite product, we recall that a basis for the product topology is given by sets of the form $U \times V$ where U and V are both open in X.

Suppose X is Hausdorff. We will show that the diagonal Δ is closed by showing that $(X \times X) \setminus \Delta$ is open. To see this, it suffices to show that for any (x, y) with $x \neq y$, there is an open subset in $X \times X$ containing (x, y) which does not intersect Δ. (The compliment of Δ is then the union of such open sets, which must therefore be open.) Fix a pair (x, y) with $x \neq y$. Since X is Hausdorff, there are disjoint open sets U and V so that $x \in U$ and $y \in V$. Observe that disjointness implies that $U \times V$ does not intersect Δ. In summary, we have shown that (x, y) lies in the open set $U \times V$ which is contained in $X \setminus \Delta$.

Conversely, suppose that $(X \times X) \setminus \Delta$ is open. Let $(x, y) \in (X \times X) \setminus \Delta$. Then since the sets of the form $U \times V$ with $U, V \subset X$ open form a basis for the topology, we know that $(X \times X) \setminus \Delta$ is a union of such sets. It follows that there is a pair of open sets $U \times V$ so that $(x, y) \in U \times V$ and $U \times V \subset (X \times X) \setminus \Delta$ is open. We conclude that $U \times V$ is disjoint from the diagonal, which is the same as saying that $U \cap V = \emptyset$. Also $(x, y) \in U \times V$ is the same as $x \in U$ and $y \in V$. This verifies that X is Hausdorff.

(b) Show that a product of Hausdorff spaces is Hausdorff.

Solution: Suppose $\{X_i \ : \ i \in \Lambda\}$ is a collection of Hausdorff topological spaces, and endow $X = \prod_{i \in \Lambda} X_i$ with the product topology. Let $x \mapsto x_i$ denote the projection $X \to X_i$, which is continuous. (Continuity of these maps defines the topology.) Let $x, y \in X$ be distinct. Then, there is some $j \in \Lambda$ so that $x_j \neq y_j$. Then because the space X_j is Hausdorff, there are disjoint open sets $U, V \subset X_j$ so that $x_j \in U$ and $y_j \in V$. By continuity of the map π_j, the sets $\pi_j^{-1}(U)$ and $\pi_j^{-1}(V)$ are open. They are also disjoint sets since their images under the map π_j are disjoint. Further, $x \in \pi_j^{-1}(U)$ and $y \in \pi_j^{-1}(V)$, which verifies that X is Hausdorff.

5. *(Lang II.5.5c)* Let X be a metric space. For each $x \in X$, define the function f_x on X by $f_x(y) = d(x, y)$. Let $\| \cdot \|$ be the sup norm.

(a) Show that $d(x, y) = \|f_x - f_y\|$.

Solution: By definition,

$$\|f_x - f_y\| = \sup \ \{|d(x, z) - d(y, z)| \ : \ z \in X\}.$$

By taking $z = y$, we observe that

$$\|f_x - f_y\| \geq |d(x, y) - d(y, y)| = d(x, y).$$

Now let $z \in X$ be arbitrary. We will show that $|d(x, z) - d(y, z)| \leq d(x, y)$ for every z. From this it follows that $\|f_x - f_y\| \leq d(x, y)$ as required. Fix $z \in X$. Observe that we have the triangle inequality, $d(x, y) + d(y, z) \geq d(x, z)$. It follows that

$$d(x, z) - d(y, z) \leq d(x, y).$$

We also have the triangle inequality, $d(y, x) + d(x, z) \geq d(y, z)$. From this it follows that

$$d(x, z) - d(y, z) \geq -d(x, y).$$

Taken together, we see that $|d(x, z) - d(y, z)| \leq d(x, y)$ as needed to show that $\|f_x - f_y\| \leq d(x, y)$.

(b) Let a be a fixed element of X, and let $g_x = f_x - f_a$. Show that the map $x \mapsto g_x$ is a distance-preserving embedding of X into the normed space of bounded functions on X. (*Remark:* This shows that every metric space is isometric to a subset of a normed vector space.)

Solution: First we observe that g_x is a bounded function by the prior part. Indeed, for each

$x \in X$,
$$\sup\{g_x(y) \ : \ y \in X\} = \|g_x\| = \|f_x - f_a\| = d(x, a),$$
with the last equality from the prior part. Furthermore, it is distance preserving:
$$\|g_x - g_y\| = \|(f_x - f_a) - (f_y - f_a)\| = \|f_x - f_y\| = d(x, y).$$
(It is clearly an embedding (i.e., it is injective), since if $x \neq y$, $d(x, y) > 0$ and therefore $\|g_x - g_y\| > 0$.)

6. *(Lang II.5.8ab)* Let X be a topological space and E a vector space with norm $|\cdot|$. Let $M(X, E)$ denote the set of all maps from X to E. Let $B(X, E)$ denote the set of bounded maps from X to E endowed with the sup norm defined by $\|f\| = \sup\{|f(x)| \ : \ x \in X\}$. Let $BC(X, E) \subset B(X, E)$ be the set of bounded continuous maps.

   (a) Show that $BC(X, E)$ is closed in $B(X, E)$.

   Solution: Observe that $B(X, E)$ is a normed vector space. So, to show $BC(X, E)$ is closed it suffices to prove that given any sequence $f_n \in BC(X, E)$ converging to $f \in B(X, E)$, then f is actually continuous.

   (**Remark:** A sequence of functions $\{f_n\}$ converges to f *uniformly* if it converges to f in the sup norm $\|\cdot\|$ as in the statement of the problem. Thus, we are proving a general form of the theorem "a uniform limit of continuous functions is continuous.")

   Suppose that $\{f_n \in BC(X, E)\}$ converges to $f \in B(X, E)$. To show f is continuous, it suffices to prove it is continuous at all points of X. So we will show that for all $x \in X$ and all $\epsilon > 0$ there is a neighborhood U of x so that $|f(x) - f(u)| < \epsilon$ for all $u \in U$. Pick $x \in X$ and $\epsilon > 0$. Then since $f_n \to f$, there is an N so that $|f_N - f| < \frac{\epsilon}{3}$. In other words,
$$\sup \ \{|f_N(x) - f(x)| \ : \ x \in X\} < \frac{\epsilon}{3}.$$
   Also by continuity of f_N, there is a neighborhood U of x so that $u \in U$ implies $|f_N(u) - f_N(x)| < \frac{\epsilon}{3}$. Then for $u \in U$, we have:
$$|f_N(u) - f_N(x)| < \frac{\epsilon}{3}. \quad |f_N(x) - f(x)| < \frac{\epsilon}{3}. \quad |f_N(u) - f(u)| < \frac{\epsilon}{3}.$$
   By use of the triangle inequality, we see that for $u \in U$,
$$|f(x) - f(u)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(u)| + |f_N(u) - f(u)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$
   Thus, f is continuous at x. Since x was arbitrary, f is continuous.

   (b) A *Banach space* is a complete normed vector space. Show that if E is a Banach space, then $B(X, E)$ is complete.

   Solution: Let $\{f_n\}$ be a Cauchy sequence in $B(X, E)$. We claim that it follows that for any $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in E. This uses the definition of Cauchy sequence. Let $x \in X$ be arbitrary. To show $\{f_n(x)\}$ is Cauchy, we will show that for all $\epsilon > 0$, there is an N so that $n, m > N$ implies $|f_n(x) - f_m(x)| < \epsilon$. Fix some $\epsilon > 0$. Since $\{f_n\}$ is Cauchy, there is an N so that $n, m > N$ implies $\|f_n - f_m\| < \epsilon$. So by definition of the sup norm, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$ as desired.

Now since E is a Banach space and for each $x \in X$, the sequence $\{f_n(x)\}$ is Cauchy, there is a limit which we define to be $f(x) = \lim_{n\to\infty} f_n(x)$. This defines a function $f : X \to E$.

We claim that $\{f_n\}$ converges to this new function f in the sup norm (or uniform) topology. Let $\epsilon > 0$. We need to show that there is an N so that $n > N$ implies $\|f_n - f\| < \epsilon$. Since $\{f_n\}$ is Cauchy, we can define N so that $n, m > N$ implies that $\|f_n - f_m\| < \frac{\epsilon}{2}$. Fix $n > N$. Then because $f(x)$ is the limit of $f_m(x)$ as $m \to \infty$, for any $x \in X$ we have

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)|,$$

and thus $|f_n(x) - f(x)| \leq \frac{\epsilon}{2}$ because $n > N$ and by the above remarks. In particular, since x was arbitrary (and N did not depend on x), $n > N$ implies $|f_n - f| < \epsilon$, which verifies the definition of convergence.

It remains to show that f is bounded. Since f_n tends to f, there is an n so that $|f_n - f| < 1$. Then $|f_n(x) - f(x)| < 1$ for all $x \in X$. Then, by the triangle inequality, for all $x \in X$,

$$|f(x)| \leq |f_n(x)| + |f_n(x) - f(x)| < |f_n(x)| + 1 < |f_n| + 1 < \infty,$$

since $f_n \in B(X, E)$.

7. Let X be a topological space. Then, X is called *separable* if it has a countable base (or basis) for its topology. A set $A \subset X$ is *dense (in X)* if its closure $\bar{A} = X$.

(a) *(Lang II.15)* Show that a separable space has a countable dense subset.

Solution: **Remark:** Recall that $X \smallsetminus \bar{A} = \text{Int}(X \smallsetminus A)$. So, by definition of the interior, A is dense if and only if A intersects every open subset of X.

Suppose X is separable. Then it has a base which can be written as $\mathcal{B} = \{B_i \ : \ i \in \mathbb{N}\}$. We can assume without loss of generality that $B_1 = \emptyset$. The for each integer $i \geq 2$, $B_i \neq \emptyset$, so we can choose a point $x_i \in B_i$.

We claim that $S = \{x_i \ : \ i \geq 2\}$ is dense in X. Let $U \subset X$ be non-empty and open. We must show that there is an point from S inside of U. Since $\mathcal{B}$ is a basis, there is a subset $\Lambda \subset \mathbb{N}$ so that $U = \bigcup_{i\in\Lambda} B_i$. Then since U is non-empty, there must be an $i \in \Lambda$ with $i \geq 2$. Then $x_i \in B_i \subset U$. So S intersects U as claimed.

(b) *(Lang II.16a)* Show that if X is a metric space and has a countable dense subset, then X is separable.

Solution: Let A be a countable dense subset of X. For $x \in X$ and $r > 0$, let $B_r(x)$ denote the open ball of radius r about x. Let $\mathbb{Q}_+$ denote the positive rationals. Define

$$\mathcal{B} = \{B_r(a) \ : \ a \in A \text{ and } r \in \mathbb{Q}_+\},$$

where $B_r(a)$ denotes the open ball of radius r around $a \in X$. Then, $\mathcal{B}$ is countable because both A and $\mathbb{Q}$ are countable. ($\mathcal{B}$ is canonically the image of $A \times \mathbb{Q}_+$ under the map $(a, r) \mapsto B_r(a)$. So, it suffices to recall that $\mathbb{Q}_+$ is countable, a product of countable sets is countable, and the image of a countable set is countable.)

We claim that $\mathcal{B}$ is a basis for the metric topology. That is, we need to show that every open set in U is a union of elements of $\mathcal{B}$. Since the collection of all balls forms a basis for the metric topology, it suffices to show that for any open ball $U \subset X$ is the union of elements of $\mathcal{B}$. Let $x_0 \in X$ and let $r_0 > 0$ be a real number, and define U to be the open ball centered at x_0 of radius r_0. To prove this it suffices to find for each $x \in U$ an element $V_x \in \mathcal{B}$ so that $x \in V_x$ and $V_x \subset U$, because then $U = \bigcup_{x \in U} V_x$. Let $x \in U$. Then $d(x, x_0) < r_0$. Define

$$\epsilon = \frac{1}{2}\big(r_0 - d(x, x_0)\big) > 0.$$

Let W be the open ball of radius ϵ about x. Then, by density of A there is a point $a \in A \cap W$. Observe that $d(a, x) < \epsilon$. Since the rationals are dense in $\mathbb{R}$, there is a rational r satisfying $d(a, x) < r < \epsilon$. Observe that by construction, $B_r(a) \in \mathcal{B}$ and $x \in B_r(a)$. We also claim that $B_r(a) \subset U$. To see this let $y \in B_r(a)$. Because of our choice of U, it suffices to prove that $d(y, x_0) < r_0$. By the triangle inequality, we have

$$d(y, x_0) \leq d(y, a) + d(a, x) + d(x, x_0) < r + r + d(x, x_0) < 2\epsilon + d(x, x_0) = r_0.$$

8. (Lang II.5.17). An *open covering* of a topological space X is a collection $\mathcal{U}$ of open sets so that $X = \bigcup_{U \in \mathcal{U}} U$. A *subcover* is a subset $\mathcal{V} \subset \mathcal{U}$ which is still a cover (i.e., $X = \bigcup_{V \in \mathcal{V}} V$).

   (a) Show that every open covering of a separable space has a countable subcovering.

Solution: Suppose X is separable. Then it has a countable basis $\mathcal{B} = \{B_i \ : \ i \in \mathbb{N}\}$. Let $\mathcal{U}$ be an open covering.

Define $J \subset \mathbb{N}$ to be the collection of all $i \in \mathbb{N}$ so that there is a $U \in \mathcal{U}$ with $B_i \subset U$. Then, for each $j \in J$, we can choose a $U_j \in \mathcal{U}$ with $B_j \subset U_j$. We claim that $\{U_j \ : \ j \in J\}$ is a countable subcover. It is clearly countable since any subset of the naturals is countable, and any image of a countable set is countable. It remains to prove that $\{U_j \ : \ j \in J\}$ covers X. Let $x \in X$. Then since $\mathcal{U}$ is a covering, there is a $U \in \mathcal{U}$ so that $x \in U$. Then because $\mathcal{B}$ is a basis, there is an $I \in \mathbb{N}$ so that $x \in B_i$ and $B_i \subset U$. But then by definition of J, we have $i \in J$. So, in particular, $x \in B_i \subset U_i$ and $i \in J$. Since x was arbitrary $X \subset \bigcup_{j \in J} U_j$.

   (b) Show that a disjoint collection of open sets in a separable space is countable.

Solution: Suppose X is separable. Then it has a countable basis $\mathcal{B} = \{B_i \ : \ i \in \mathbb{N}\}$. Let $\mathcal{V}$ be a collection of disjoint open sets.

Similar to the prior part, let $J \subset \mathbb{N}$ be the collection of all $i \in \mathbb{N}$ so that $B_i \neq \emptyset$ and there is a $V \in \mathcal{V}$ with $B_i \subset V$. Observe that J is countable. We claim that for each $j \in J$, there is a unique $V \in \mathcal{V}$ so that $B_j \subset V$. To see this suppose $B_j \subset V$ and $B_j \subset V'$ with $V, V' \in \mathcal{V}$. Then, $B_j \subset V \cap V'$ and since the sets in $\mathcal{V}$ are disjoint $V = V'$, which proves uniqueness. So, for $j \in J$, we define $V_j \in \mathcal{V}$ so that $B_j \subset V_j$. This defines a map $\psi : J \to \mathcal{V}$ via $j \mapsto V_j$. We claim that the image of this map contains all non-empty elements of $\mathcal{V}$. From this it follows that $\mathcal{V}$ is countable. (Any image of a countable set is countable, and a union of countable sets is countable.) To see surjectivity recall that $\mathcal{B}$ is a basis. In particular, every non-empty $V \in \mathcal{V}$ is a union of elements of $\mathcal{B}$. If B_j is a non-empty element in this union, then $V = V_j$.

(c) Show that a base (or basis) for the topology of a separable space contains a countable base for the topology. (The red words were added to clarify the question.)

Solution: Suppose X is separable. Then it has a countable basis, which we may write as $\mathcal{C} = \{C_i : i \in \mathbb{N}\}$. Now let $\mathcal{B}$ be another basis. We will show that there is a countable subset $\mathcal{S} \subset \mathcal{B}$ which is also a basis. That is, we need to choose a countable $\mathcal{S} \subset \mathcal{B}$ and show that every open set is a union of elements of $\mathcal{S}$. Since $\mathcal{C}$ is also a basis, it will suffice to show that each C_i is a union of elements of $\mathcal{S}$. It is sufficient therefore to show that for each $i \in \mathbb{N}$, there is a countable $\mathcal{B}_i \subset \mathcal{B}$ so that

$$C_i = \bigcup_{U \in \mathcal{B}_i} U.$$

Indeed, then $\mathcal{S} = \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$ is a basis, and is countable because a countable union of countable sets is countable.

Let $i \in \mathbb{N}$ be arbitrary. It remains to show that there is a countable $\mathcal{B}_i \subset \mathcal{B}$ so that $C_i = \bigcup_{U \in \mathcal{B}_i} U$. Since $\mathcal{B}$ is a basis, we can choose a collection $\mathcal{D} \subset \mathcal{B}$ so that $C_i = \bigcup_{D \in \mathcal{D}} D$. Then because $\mathcal{C}$ is a basis, for each $D \in \mathcal{D}$, there is a subset $J_D \subset \mathbb{N}$ so that $D = \bigcup_{j \in J_D} C_j$. Let $J = \bigcup_{D \in \mathcal{D}} J_D \subset \mathbb{N}$. Then observe that

$$\bigcup_{j \in J} C_j = \bigcup_{D \in \mathcal{D}} \bigcup_{j \in J_D} C_j = \bigcup_{D \in \mathcal{D}} D = C_i.$$

Now observe that for any $j \in J$, we have $j \in J_D$ for some $D \in \mathcal{D}$. In particular then $C_j \subset D$. So for each $j \in J$, we can choose some $D_j \in \mathcal{D}$ so that $C_j \subset D_j$. Thus, we have defined a map $J \to \mathcal{D}$ by $j \mapsto D_j$. We define $\mathcal{B}_i$ to be the image of this map; $\mathcal{B}_i = \{D_j : j \in J\}$. Since J is countable, we see that $\mathcal{B}_i$ is countable. We need to show that the union of elements of $\mathcal{B}_i$ gives C_i. Clearly since $\mathcal{B}_i \subset \mathcal{D}$,

$$\bigcup_{D \in \mathcal{B}_i} D \subset \bigcup_{D \in \mathcal{D}} D = C_i.$$

For the reverse inclusion, observe that $C_j \subset D_j$ for $j \in J$ so

$$C_i = \bigcup_{j \in J} C_j \subset \bigcup_{j \in J} D_j = \bigcup_{D \in \mathcal{B}_i} D.$$

Thus, $C_i = \bigcup_{D \in \mathcal{B}_i} D$ as required.