Math 70100: Functions of a Real Variable I
Homework 12, due Wednesday, December 10th.

We let L^1 denote the collection space of integrable functions from $\mathbb{R} \to \mathbb{R}$.

1. *(Folland §2.3 # 20)* (A generalized Dominated Convergence Theorem) Let $f, g \in L^1$ and let $\{f_n\}$ and $\{g_n\}$ be sequences in L^1. Show that if $|f_n| \leq g_n$, $f_n \to f$ and $g_n \to g$ pointwise a.e., and $\lim_{n\to\infty} \int g_n = \int g$, then $\int f = \lim_{n\to\infty} \int f_n$. (*Hint:* Rework the proof of the dominated convergence theorem, either in class or following Folland. *Remark:* The condition that $\int g_n \to \int g$ is necessary as the example $g_n = \frac{1}{n}\chi_{[1,n]}$ illustrates.)

Solution: Since $-g_n \leq f_n \leq g_n$, we observe that $g_n + f_n \geq 0$ and $g_n - f_n \geq 0$. The sequence of functions in N given by

$$h_N = \inf_{n \geq N} g_n + f_n$$

is therefore non-negative and increases pointwise to $g + f$ almost everywhere. Observe that h_N is measurable and dominated by the integrable function $g + f$. Therefore, h_N is integrable. Then by the monotone convergence theorem,

$$\int (g + f) = \lim_{N\to\infty} \int \inf_{n\geq N} (g_n + f_n).$$

Similar remarks about the sequence $g_n - f_n$ indicate that

$$\int (g - f) = \lim_{N\to\infty} \int \inf_{n\geq N} (g_n - f_n).$$

Observe that we have

$$\inf_{n\geq N} (g_n + f_n) \leq g_N + f_N \quad \text{and} \quad \inf_{n\geq N} (g_n - f_n) \leq g_N - f_N.$$

It follows that

$$-g_N + \inf_{n\geq N} (g_n + f_n) \leq f_N \leq g_N - \inf_{n\geq N} (g_n - f_n).$$

Integrating everything yields:

$$-\int g_N + \int \inf_{n\geq N} (g_n + f_n) \leq \int f_N \leq \int g_N - \int \inf_{n\geq N} (g_n - f_n).$$

Observe that the limit of the left hands side is

$$\lim_{N\to\infty} \left[-\int g_N + \int \inf_{n\geq N} (g_n + f_n) \right] = -\int g + \int (g + f) = \int f.$$

Similarly, the limit of the right hand side is

$$\lim_{N\to\infty} \left[\int g_N - \int \inf_{n\geq N} (g_n - f_n) \right] = \int g - \int (g - f) = \int f.$$

So, by the squeeze theorem, we see $\lim_{N\to\infty} f_N = \int f$.

2. *(Folland §2.3 # 21)* Suppose $\{f_n \in L^1\}$ is a sequence of functions converging to $f \in L^1$ pointwise a.e., then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. (*Hint:* Use the prior exercise. *Remark:* All the functions need to be integrable here.)

> **Solution:** First suppose that $\lim_{n\to\infty} \int |f_n - f| \to 0$. Observe that $-|f_n - f| \le |f_n| - |f| \le |f_n - f|$ pointwise. So, for every n,
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> $$-\int |f_n - f| \le \int |f_n| - |f| \le \int |f_n - f|.$$
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> By hypothesis, the left and right sides tend to zero as $n \to \infty$, so by the squeeze theorem $\lim_{n\to\infty} \int |f_n| - |f| = 0$. This is equivalent to saying $\int |f_n| \to \int |f|$.
>
> Now suppose that $\int |f_n| \to \int |f|$. Set $\phi_n = |f_n - f|$ and $\psi_n = |f_n| + |f|$. Clearly $\phi_n \le \psi_n$ everywhere. By our hypothesis that $f_n \to f$ pointwise a.e., we know that $\phi_n \to 0$ and $\psi_n \to 2|f|$ pointwise a.e.. Also, because $\int |f_n| \to \int |f|$, we know that $\int \psi_n \to \int 2|f|$. We conclude from the prior part that $\int \phi_n \to \int 0 = 0$, or equivalently, $\lim_{n\to\infty} \int |f_n - f| = 0$.

3. *(Folland §2.3 # 19b)* Find a sequence of integrable functions $f_n : \mathbb{R} \to [0, \infty)$ so that $\{f_n\}$ converges uniformly to $f : \mathbb{R} \to [0, \infty)$ but f is not integrable.

> **Solution:** Let $f_n(x) = \frac{1}{x}\chi_{[1,n]}(x)$, which is integrable because it is discontinuous at only two points, is bounded, and compactly supported. Let $f = \frac{1}{x}\chi_{[1,\infty)}(x)$. Then
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> $$f - f_n = \frac{1}{x}\chi_{(n,\infty)},$$
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> which is bounded in absolute value by $\frac{1}{n}$ which tends to zero. Thus $f_n \to f$ uniformly. Furthermore,
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> $$\int f_n = \int_1^n \frac{1}{x}\, dx = \ln n.$$
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> Since $f_n \le f$ for all n, we see that $\int f \ge \ln n$ for all n. Thus $\int f = +\infty$, and f is not integrable.

4. *(Folland §2.3 # 26)* Show that if $f \in L^1$ and $F(x) = \int_{-\infty}^x f(x)\, d\lambda(x)$, then $F : \mathbb{R} \to \mathbb{R}$ is continuous.

> **Solution:** First we will show it for the case where $f : \mathbb{R} \to [0, \infty)$ is integrable. For $a < b$ define the strip
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> $$S_{a,b} = \{(x, y) \ : \ a \le x < b\}.$$
>
> Observe that if $a < b$, then
>
> $$F(b) - F(a) = \lambda_2\big(\mathcal{U}f \cap S_{a,b}\big).$$

Fix a, and observe that

$$\bigcap_{b>a}(\mathcal{U}f \cap S_{a,b}) = \mathcal{U}f \cap \{(x,y) \ : \ x = a\}.$$

So by measure continuity, we have

$$\lim_{b\to a^+} F(b) - F(a) = \lambda_2\big(\mathcal{U}f \cap \{(x,y) \ : \ x = a\}\big) = 0.$$

Similarly, fixing b, we see

$$\bigcap_{a<b}(\mathcal{U}f \cap S_{a,b}) = \emptyset$$

and so by measure continuity

$$\lim_{a\to b^-} F(b) - F(a) = \lambda_2(\emptyset) = 0.$$

By combining these statements, we conclude that $\lim_{y\to x} F(y) = F(x)$.

Now consider the statement for $f : \mathbb{R} \to \mathbb{R}$ integrable. Consider the positive and negative parts f_+ and f_-, and recall that both are positive functions, $f = f_+ - f_-$ and $\int f = \int f_+ - \int f_-$. Then, by applying the observations above,

$$\lim_{y\to x} F(y) = \lim_{y\to x} \int_{-\infty}^{y} f_+ - \int_{-\infty}^{y} f_- = \int_{-\infty}^{x} f_+ - \int_{-\infty}^{x} f_- = F(x).$$

5. *(Pugh Chapter 6 # 60a)* Let $E \subset \mathbb{R}$ be a measurable set having positive Lebesgue measure. Prove Steinhaus' Theorem: E meets its t-translates for all sufficiently small $t \in \mathbb{R}$. (*Hint:* density points.)

Solution: Let p be a density point for E. Then we can choose a ball $B = (p-r, p+r)$ of radius r centered at p so that

$$\lambda(B \cap E) > \frac{2}{3}\lambda(B) = \frac{4r}{3}.$$

Now suppose $t \in \mathbb{R}$ and $0 \le t < \frac{r}{3}$. Let $C = B \setminus E$. It then follows that $\lambda(C) \le \frac{2r}{3}$. We will show that $E \cap (E+t) \ne \emptyset$ by showing that $(E \cap B) \cap \big((E \cap B) + t\big) \ne \emptyset$. Here $+t$ denotes translation by t. Observe that both sets $A = E \cap B$ and $B = (E \cap B) + t$ are contained in the interval $I = (p-r, p+r+t)$, and the measure of their compliments is

$$\lambda(I \setminus A) = \lambda(I \setminus B) = \lambda(C) + t < r.$$

It follows that $\lambda\big(I \setminus (A \cup B)\big) < 2r$. Therefore,

$$\lambda(A \cap B) = \lambda(I) - \lambda\big(I \setminus (A \cup B)\big) > (2r + t) - 2r = t \ge 0.$$

Since $\lambda(A \cap B)$ measure is positive, $A \cap B \ne \emptyset$ and hence $E \cap (E+t) \ne \emptyset$.

We proved the result for small non-negative t. By reflecting E in the origin, we see that it also holds for small non-positive t.

6. *(Zakeri's Homework 11 # 4)* Let $f \in L^1$, and let E be a Lebesgue measurable set of positive measure. The average value of f on E is

$$A(f;E) = \frac{1}{\lambda(E)} \int_E f.$$

Prove that if $A(f;E) \in [a,b]$ for every such E, then $f(x) \in [a,b]$ for almost every x.

Solution: Consider the set

$$A = \{x \in \mathbb{R} \ : \ f(x) \notin [a,b]\},$$

which is measurable because $f \in L^1$. We prove the contrapositive. We will show that if $\lambda(A) > 0$, then there is a set E of positive measure so that $A(f;E) \notin [a,b]$.

Observe that A is naturally the union of two disjoint measurable sets, $A = A_+ \sqcup A_-$ with

$$A_+ = \{x \in \mathbb{R} \ : \ f(x) > b\} \quad \text{and} \quad A_- = \{x \in \mathbb{R} \ : \ f(x) < a\}.$$

Observe that $\lambda(A_+) + \lambda(A_-) = \lambda(A) > 0$.

Assume that $\lambda(A_+) > 0$. (It could be that $\lambda(A_+) = \infty$.) Observe that $A_+ = \bigcup_n \big(A_+ \cap [-n,n]\big)$. Measure continuity then tells us that $\lambda(A_+) = \lim_{n\to\infty} \lambda\big(A_+ \cap [-n,n]\big)$. Then there is some n so that the set $E = A_+ \cap [-n,n]$ satisfies $0 < \lambda(E) < \infty$. We claim that this choice works. We will use the fact that when the integral of a non-negative function is zero, the function must be zero almost everywhere. Since $f - b$ is a positive function on E, it must have positive integral. Therefore

$$\int_E (f-b) > 0 \quad \text{and thus} \int_E f > b\lambda(E).$$

We have shown that $A(f;E) = \frac{1}{\lambda(E)} \int_E f > b$.

The same argument works for the case when $\lambda(A_-) > 0$.

7. *(Spring 2013 Qual)* Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[0,1]$, and satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ with $f(0) = 1$. Prove that $f(x) = e^{ax}$ for some constant $a \in \mathbb{R}$.

Solution: First observe that f never non-positive values. It can't take the value zero because if $f(x) = 0$ then

$$1 = f(x-x) = f(x)f(-x) = 0,$$

which is a contradiction. Similarly, f cannot take negative values because $f(x) = f(\frac{x}{2})^2 \geq 0$.

The other important observation is that for every $\frac{p}{q} \in \mathbb{Q}$ and each $x \in \mathbb{R}$, we have

$$f(\frac{p}{q}x) = f(x)^{\frac{p}{q}}.$$

This further implies that if $f(x) = e^{ax}$ then $f(\frac{p}{q}x) = e^{a(p/qx)}$ for all rationals $\frac{p}{q}$, which will be what we use in the proof. To see this, first observe it holds for natural numbers n: $f(nx) = f(x)^n$. For positive rationals $\frac{p}{q}$, observe that the above implies $f(px) = f(x)^p$ and $f(px)^q = f(\frac{px}{q})$. By combining these expressions, we get the above result for positive rationals. The equation also clearly holds when $\frac{p}{q} = 0$ since $f(0) = 1$. We can extend to the negative rationals by observing that $f(x)f(-x) = 1$.

Since $f(1/2)$ is positive, there is a unique real number so that $f(1/2) = e^{a/2}$. Now suppose that there is an x_0 so that $f(x_0) \neq e^{ax_0}$. We have $f(x_0) = e^{bx_0}$ for some $b \neq a$. We will derive a contradiction to this. By considering rational multiples of x_0, we obtain as sequence of real numbers $\{x_n = \frac{p_n x_0}{q_n}\}$ converging to $1/2$ and satisfying $f(x_n) = e^{bx_n}$. We will derive a contradiction from this.

Let $I \subset (0, \frac{1}{2})$ be an closed interval of positive length (whose endpoints are bounded away from 0 and $\frac{1}{2}$). Let J be I translated right by adding $\frac{1}{2}$. Observe that there is a relationship between the integrals:

$$\int_J f(x)\ dx = \int_I f(x + \frac{1}{2}) = \int_I e^{a/2} f(x)\ dx = e^{a/2} \int_I f(x)\ dx. \tag{1}$$

Similarly, we can let J_n denote the translation of I by x_n. Since x_n tends to $\frac{1}{2}$, for sufficiently large n, $J_n \subset [0, 1]$, so f is integrable on J_n. For similar reasons, we have

$$\int_{J_n} f(x)\ dx = \int_I f(x + x_n)\ dx = e^{bx_n} \int_I f(x)\ dx.$$

Observe that as $n \to \infty$, we have $J_n \to J$, in the sense that the endpoints converge. Then by continuity of the integral (anti-derivative) we have

$$\int_J f(x)\ dx = \lim_{n\to\infty} \int_{J_n} f(x)\ dx = \lim_{n\to\infty} e^{bx_n} \int_I f(x)\ dx = e^{b/2} \int_I f(x)\ dx.$$

Since $f > 0$, we know $\int_I f > 0$. Thus the equation above and equation 1 together imply that $a = b$, which is a contradiction.

Remark: We are using the fact that if f is an integrable function on an interval containing 0, then the anti-derivative

$$g(t) = \int_0^t f(x)\ dx$$

is continuous. This implies continuity of integrals over intervals, since for $[a_n, b_n] \to [a, b]$, we have

$$\int_{[a,b]} f = g(b) - g(a) = \lim_{n\to\infty} g(b_n) - g(a_n) = \lim_{n\to\infty} \int_{[a_n,b_n]} f.$$

Further source of help: There is a document discussing this problem here: