Math 70100: Functions of a Real Variable I
Homework 12, due Wednesday, December 10th.

We let L^1 denote the collection space of integrable functions from $\mathbb{R} \to \mathbb{R}$.

1. *(Folland §2.3 # 20)* (A generalized Dominated Convergence Theorem) Let $f, g \in L^1$ and let $\{f_n\}$ and $\{g_n\}$ be sequences in L^1. Show that if $|f_n| \leq g_n$, $f_n \to f$ and $g_n \to g$ pointwise a.e., and $\lim_{n\to\infty} \int g_n = \int g$, then $\int f = \lim_{n\to\infty} \int f_n$. (*Hint:* Rework the proof of the dominated convergence theorem, either in class or following Folland. *Remark:* The condition that $\int g_n \to \int g$ is necessary as the example $g_n = \frac{1}{n}\chi_{[1,n]}$ illustrates.)

2. *(Folland §2.3 # 21)* Suppose $\{f_n \in L^1\}$ is a sequence of functions converging to $f \in L^1$ pointwise a.e., then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. (*Hint:* Use the prior exercise. *Remark:* All the functions need to be integrable here.)

3. *(Folland §2.3 # 19b)* Find a sequence of integrable functions $f_n : \mathbb{R} \to [0, \infty)$ so that $\{f_n\}$ converges uniformly to $f : \mathbb{R} \to [0, \infty)$ but f is not integrable.

4. *(Folland §2.3 # 26)* Show that if $f \in L^1$ and $F(x) = \int_{-\infty}^{x} f(t)\ d\lambda(t)$, then $F : \mathbb{R} \to \mathbb{R}$ is continuous.

5. *(Pugh Chapter 6 # 60a)* Let $E \subset \mathbb{R}$ be a measurable set having positive Lebesgue measure. Prove Steinhaus' Theorem: E meets its t-translates for all sufficiently small $t \in \mathbb{R}$. (*Hint:* density points.)

6. *(Zakeri's Homework 11 # 4)* Let $f \in L^1$, and let E be a Lebesgue measurable set of positive measure. The average value of f on E is

$$A(f; E) = \frac{1}{\lambda(E)} \int_E f.$$

   Prove that if $A(f; E) \in [a, b]$ for every such E, then $f(x) \in [a, b]$ for almost every x.

7. *(Spring 2013 Qual)* Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[0, 1]$, and satisfies $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ with $f(0) = 1$. Prove that $f(x) = e^{ax}$ for some constant $a \in \mathbb{R}$.