Math 70100: Functions of a Real Variable I
Homework 11, due Wednesday, November 26th.

1. *(Modified from Pugh, Chapter 6 # 28)* A non-negative linear combination of measurable characteristic functions is a *simple function* (or *step function*). That is, a simple function has the form
$$\phi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$$
where $E_1, \ldots, E_n$ are measurable sets and $c_1, \ldots, c_n$ are non-negative constants. (The *characteristic function* of $E \subset \mathbb{R}$ is the function $\chi_E : \mathbb{R} \to \{0, 1\}$ so that $\chi_E(x) = 1$ if and only if $x \in E$.) We say that $\sum c_i \chi_{E_i}$ *expresses* ϕ. If the E_i are disjoint and non-empty and the c_i are distinct and positive, then the expression for ϕ is called *canonical*.

   (a) Show that a canonical expression for a simple function exists and is unique. (*Remark: It might be useful to review part (b) to see if you want to prove more here.*)

   Solution: Let $\phi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$ be a simple function. Observe that ϕ takes no more than 2^n values: the value $\phi(x)$ is determined by the set of i so that $x \in E_i$. Let Y denote the finite set of values taken by ϕ. Then observe that
   $$\phi = \sum_{y \in Y} y \chi_{\phi^{-1}(\{y\})}.$$
   This is a canonical expression for ϕ.

   To see it is unique, suppose $\sum_{j \in J} d_j \chi_{F_j}$ is a canonical expression for some functions ψ. We will show that if the set of pairs $P_\psi = \{(F_j, d_j)\}$ is distinct from the set of pairs $P_\phi = \{(\phi^{-1}(y), y) \ : \ y \in Y\}$ then $\phi \neq \psi$. First suppose there is a j so that $(F_j, d_j) \notin P_\phi$. If $d_j \notin Y$, then as F_j is non-empty, there is an $x \in F_j$ and $\phi(x) \in Y$ is different than $\psi(x) = d_j$. Now suppose $d_j = y$ for some $y \in Y$ but $F_j \neq \phi^{-1}(y)$. Then there is an $x \in F_j \smallsetminus \phi^{-1}(y)$ or an $x \in \phi^{-1}(y) \smallsetminus F_j$. In either case, we see that $\phi(x) \neq \psi(x)$. This proves that $P_\psi \not\subset P_\phi$ implies $\psi \neq \psi$. The same argument shows that if $P_\phi \not\subset P_\psi$ implies $\phi \neq \psi$.

   (b) If ϕ is a simple function with canonical representation $\sum_{i=1}^{n} c_i \chi_{E_i}$, define the "integral" $I(\phi) = \sum_i c_i \lambda(E_i)$. Show that if $\sum_{j=1}^{m} d_j \chi_{F_j}$ is a (not-necessarily canonical) expression of ϕ, then
   $$I(\phi) = \sum_{j=1}^{n} d_j \lambda(F_j).$$

   Solution: Let $J = \{1, \ldots, m\}$ which indexes the non-canonical expression for ϕ. Let $A \subset J$ and define $F_A = \bigcap_{j \in A} F_j$. The sets F_A are indexed by the power set of J, are disjoint and measurable and cover the set of non-zero values of ϕ. Observe that for $x \in F_A$,
   $$\phi(x) = \sum_{j \in A} d_j = c_i \quad \text{for some } i.$$

Define $c_A = c_i$, where c_i is determined as above. The fact that our non-canonical expression yields ϕ on E_i tells us that

$$E_i = \bigsqcup_{A \,:\, c_A = c_i} F_A.$$

Therefore, it follows that

$$\lambda(E_i) = \sum_{A \,:\, c_A = c_i} \lambda(F_A).$$

Every A with $c_A \neq 0$ is thus accounted for by some i. Since $c_i = c_A$ when $F_A \subset E_i$, plugging this into our expression for the integral yields

$$I(\phi) = \sum_i c_i \lambda(E_i) = \sum_{A \subset J} c_A \lambda(F_A).$$

Because our expressions for ϕ coincide on F_A, we see that $c_A = \sum_{j \in A} d_j$. So, we have

$$I(\phi) = \sum_{A \subset J} \sum_{j \in A} d_j \lambda(F_A).$$

This is a finite sum, so we can rearrange it as:

$$I(\phi) = \sum_{j \in J} \left(d_j \sum_{A \subset J \,:\, j \in A} \lambda(F_A) \right).$$

Now since each F_j is a disjoint union of the F_A where $j \in A$, we see by finite additivity of measure,

$$I(\phi) = \sum_{j \in J} d_j \lambda(F_j),$$

which is the expression sought.

(c) Infer from (b) that the map I from simple functions to $\mathbb{R}$ given by $\phi \mapsto I(\phi)$ is linear.

Solution: Suppose $a \in \mathbb{R}$ and $\phi = \sum_i c_i \chi_{E_i}$. Observe that $a\phi$ also a simple function and has expression

$$a\phi = \sum_i a c_i \chi_{E_i}.$$

Thus,

$$I(a\phi) = \sum_i a c_i \lambda(E_i) = a \sum_i c_i \lambda(E_i) = a I(\phi).$$

Now suppose $\phi = \sum_i c_i \chi_{E_i}$ and $\psi = \sum_j d_j \chi_{F_j}$. Then,

$$\sum_i c_i \chi_{E_i} + \sum_j d_j \chi_{F_j}$$

is an expression of $\phi + \psi$. We conclude that the sum satisfies:

$$I(\phi + \psi) = \sum_i c_i \lambda(E_i) + \sum_j d_j \lambda(F_j) = I(\phi) + I(\psi).$$

(d) Given a measurable function $f : \mathbb{R} \to [0, \infty)$, show there exists a pointwise increasing sequence of simple functions $\{\phi_n\}$ whose pointwise limit is f.

Solution: For a natural number n, let

$$Y_n = \{\frac{p}{2^n} \ : \ p \in \mathbb{N} \text{ and } \frac{p}{2^n} \in [0, n].\}.$$

Then define

$$\phi_n = \sum_{y \in Y_n} y \chi_{f^{-1}\left([y, y+\frac{1}{2^n})\right)}.$$

To see that ϕ_n is increasing observe that $Y_n \subset Y_{n+1}$. The value of ϕ_n at x is the greatest value less than or equal to $f(x)$. Thus, $Y_n \subset Y_{n+1}$ implies $\phi_n(x) \leq \phi_{n+1}(x)$. Also observe that if $f(x) < n$, then

$$|f(x) - \phi_n(x)| < \frac{1}{2^n}.$$

This eventually holds for any x, so $\phi_n \to f$ pointwise.

(e) Show that for any two such sequences $\{\phi_n\}$ and $\{\psi_n\}$ increasing to f as in part (d),

$$\lim_{n\to\infty} I(\phi_n) = \lim_{n\to\infty} I(\psi_n).$$

Therefore, the definition of $I(f)$ as this limit is well-defined.

Solution: First observe that the integrals $I(\phi_n)$ and $I(\psi_n)$ are increasing. (It can be proven that $I(\phi_n) \leq I(\phi_{n+1})$ by choosing an expression for both functions using the same collection of sets E_i and comparing the formula for the integral termwise.) In particular

$$\lim_{n\to\infty} I(\phi_n) = \sup_n I(\phi_n) \quad \text{and} \quad \lim_{n\to\infty} I(\psi_n) = \sup_n I(\psi_n).$$

This suggests a path to a proof. We will show that for any n and any real number $\alpha < I(\phi_n)$, there is an m so that $I(\psi_m) \geq \alpha$. This directly implies that $\sup_n I(\psi_n) \geq \sup_n I(\phi_n)$. Our argument will be independent of the choice of sequences $\{\phi_n\}$ and $\{\psi_n\}$ so the same argument will yield $\sup_n I(\psi_n) \leq \sup_n I(\phi_n)$, so we get equality.

We will be proving that for any n and any $\alpha < I(\phi_n)$, there is an m so that $I(\psi_m) \geq \alpha$. From prior work, we can assume ϕ_n is given by a canonical expression, say

$$\phi_n = \sum_i c_i \chi_{E_i}.$$

For each $N \in \mathbb{N}$, observe that $\chi_{[-N,N]}\phi_n$ is also a step function. Indeed,

$$\chi_{[-N,N]}\phi_n = \sum_i c_i \chi_{E_i \cap [-N,N]}.$$

The sets $E_i \cap [-N, N]$ increase to E_i, so by measure continuity, $I(\chi_{[-N,N]}\phi_n) \to I(\phi_n)$. In particular, we can choose an N so that $I(\chi_{[-N,N]}\phi_n) > \alpha$. For notational convenience, define $\phi'_n = \chi_{[-N,N]}\phi_n$ and $E'_i = E_i \cap [-N, N]$.

Now we have $\alpha < I(\phi'_n) < \infty$. Let $M < \infty$ denote the maximum value taken by ϕ'_n. We can choose an $\epsilon > 0$ so that

$$\alpha < I(\phi'_n) - (2N + M)\epsilon.$$

Now consider the increasing sequence ψ_m, which converges pointwise to a function f which is pointwise larger than ψ'_n. Then for any x there is an m so that

$$\psi_m(x) > \phi'_n(x) - \epsilon.$$

Then for any m, we can consider the bad points

$$B_m = \{x \ : \ \psi_m(x) \leq \phi'_n(x) - \epsilon\}.$$

Our prior remark tells us that $\bigcap B_m = \emptyset$. Observe $B_m \subset [-N, N]$ since $\phi'_n(x) = 0$ outside $[-N, N]$, while each ϕ_m is non-negative. Thus, B_m have finite measure and so measure continuity tells us that $\lambda(B_m) \to 0$. Therefore, we can choose m so large that $\lambda(B_m) < \epsilon$. We claim that ψ_m works. In summary, by construction we have

$$\psi_m(x) \geq 0 \text{ for } x \in B_m \text{ and } \psi_m(x) > \max\{0, \phi'_n(x) - \epsilon\} \text{ for } x \in [-N, N] \setminus B_m.$$

In other words, we have the following pointwise inequality:

$$\psi_m \geq \phi'_n - \epsilon\chi_{[-N,N]\setminus B_m} - \phi'_n\chi_{B_m}.$$

Let ξ denote the expression at right. We have $I(\psi_m) \geq I(\xi)$. Now observe the following inequalities involving the terms of ξ:

$$I(\chi_{[-N,N]\setminus B_m}) \leq I(\chi_[- N, N]) = 2N.$$

$$I(\phi'_n\chi_{B_m}) \leq I(M\chi_{B_m}) = M\lambda(B_m) < M\epsilon.$$

Therefore, by linearity and the definition of ϵ, we have

$$I(\xi) = I(\phi'_n) - \epsilon I(\chi_{[-N,N]\setminus B_m}) - I(\phi'_n\chi_{B_m}) \geq I(\phi'_n) - \epsilon(2N) - M\epsilon = I(\phi'_n) - (2N+M)\epsilon > \alpha.$$

Then, $I(\psi_m) > I(\xi) > \alpha$ as required.

(f) Show that the function I from the space of measurable functions $\mathbb{R} \to [0, \infty)$ to $\mathbb{R}$ given by $f \mapsto I(f)$ is linear.

Solution: Let a and b be positive real numbers and $f, g : \mathbb{R} \to [0, \infty)$ be measurable. Let $\{\phi_n\}$ and $\{\psi_n\}$ be increasing sequences of step functions whose pointwise limits are f and g. Then $\{a\phi_n + b\psi_n\}$ is an increasing sequence of step functions converging to $af + bg$. It then follows by definition that

$$I(af + bg) = \lim_{n\to\infty} I(a\phi_n + b\psi_n).$$

Then by linearity of I action on the space of step functions, we have

$$I(af + bg) = \lim_{n\to\infty} aI(\phi_n) + bI(\psi_n) = a \lim_{n\to\infty} I(\phi_n) + b \lim_{n\to\infty} I(\psi_n) = aI(f) + bI(g).$$

2. *(Pugh, Chapter 6 # 30)* Find a sequence of measurable functions $f_n : [0, 1] \to [0, 1]$ such that $\int f_n \to 0$ as $n \to \infty$, but for no $x \in [0, 1]$ does $f_n(x)$ converge to a limit as $n \to \infty$.

Solution: Suppose $\{a_n\}$ is a sequence of real numbers with $0 < a_n \leq 1$ for all n so that $a_n \to 0$ and so that the partial sums $b_N = \sum_{n=1}^{N} a_n$ tend to infinity. (For example $a_n = \frac{1}{n}$.) Define

$$f_n(x) = \begin{cases} 1 & \text{if there is an } m \in \mathbb{Z} \text{ so that } x \in [b_n, b_{n+1}) \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\int f_n = b_{n+1} - b_n = a_n,$$

which tends to zero as $n \to \infty$ by hypothesis. Now fix x. We will show that $f_n(x)$ does not tend to zero by showing $f_n(x) = 1$ infinitely often. To see this observe that

$$[0, \infty) = \bigcup_n [b_n, b_{n+1}).$$

So, for each integer $m \geq 0$, there is an $n = n(m)$ so that $x + m \in [b_n, b_{n+1})$. Since each interval $[b_n, b_{n+1})$ has length $a_n \leq 1$, the mapping $m \mapsto n(m)$ is injective. The image of this map provides the infinitely many n for which $f_n(x) = 1$.

3. Let $\{f_n : \ n \in \mathbb{N}\}$ be a sequence of measurable functions $\mathbb{R} \to [0, \infty)$. Define $g_k = \inf_{n \geq k} f_n$, i.e.,

$$g_k(x) = \inf \ \{f_n(x) \ : \ n \geq k\}.$$

Define $h = \liminf\limits_{n\to\infty} f_n$, i.e.,

$$h(x) = \lim_{k\to\infty} g_k(x).$$

(a) Show that g_k is measurable for all $k \in \mathbb{N}$. Explain why $\int g_k \leq \int f_n$ when $n \geq k$.

Solution: To show g_k is measurable, it suffices to show that $g_k^{-1}\big((-\infty, y)\big)$ is measurable for all $y \in \mathbb{R}$. Observe $g_k^{-1}\big((-\infty, y)\big) = g_k^{-1}\big([0, y)\big)$ since g_k is non-negative.

Since $g_k = \inf_{n \geq k} f_n$, we see that $g_k(x) \in [0, y)$ if and only if there is an $n \geq k$ so that $f_n(x) \in [0, y)$. Therefore,

$$g_k^{-1}\big([0, y)\big) = \bigcup_{n \geq k} f_n^{-1}\big([0, y)\big).$$

This is a countable union of measurable sets and therefore measurable.

To see that $\int g_k \leq \int f_n$ for $n \geq k$, observe that $g_k \leq f_n$ pointwise.

(b) Prove Fatou's lemma. (Sometimes called Fatou's theorem). Prove that h is measurable and $\int h \leq \liminf_{n\to\infty} \int f_n$. (*Hint: Use the monotone convergence theorem. Remark: Sometimes Fatou's lemma is used to prove the monotone convergence theorem, though we did not do this.*)

Solution: Observe that $g_k = \inf_{n \geq k} f_n$ is increasing as k increases, since g_{k+1} is an infimum over a smaller set of functions. The monotone convergence theorem then directly implies that $h = \lim_{k\to\infty} g_k$ is measurable and that

$$\int h = \lim_{k\to\infty} \int g_k.$$

To finish the proof, recall from the previous part that $\int g_k \leq \int f_k$ for all k. Thus,

$$\int h = \liminf_{k\to\infty} \int g_k \leq \liminf_{k\to\infty} \int f_k.$$

4. *(Pugh, Chapter 6 # 55)* A sequence of measurable functions $f_n : [a, b] \to \mathbb{R}$ converges to measurable $f : [a, b] \to \mathbb{R}$ *nearly uniformly* if for every ϵ, there is a measurable set $S \subset [a, b]$ with $\lambda(S) < \epsilon$ so that $f_n \to f$ uniformly on $[a, b] \smallsetminus S$. Show that nearly uniform convergence is transitive in the following sense. Assume f_n converges to f nearly uniformly as $n \to \infty$ and that for each n there is a sequence of measurable functions $f_{n,k}$ which converges nearly uniformly to f_n as $k \to \infty$. We will show that there is a choice of a map $n \mapsto k(n)$ so that the sequence $f_{n,k(n)}$ tends to f nearly uniformly as $n \to \infty$. (All functions are measurable and defined on $[a, b]$.)

(a) Show that there is a sequence $k(n) \to \infty$ such that $f_{n,k(n)}$ converges nearly uniformly to f as $n \to \infty$.

Solution: Suppose $f_n \to f$ nearly uniformly and for all n, $f_{n,k} \to f_n$ nearly uniformly. We will begin by explaining how to produce the function $n \mapsto k(n)$.

Since $f_{n,k} \to f_n$ nearly uniformly, we can choose for each n a set S_n with $\lambda(S_n) < \frac{1}{2^n}$ and a $k = k(n)$ so that

$$|f_{n,k(n)} - f_n| < \frac{1}{2^n} \quad \text{on the set } [a, b] \smallsetminus S_n.$$

Now we prove the following:
Lemma. Suppose $f_n \to f$ uniformly on a set $[a,b] \setminus S$. Let $N \in \mathbb{N}$ and let $S' = S \cup \bigcup_{n=N}^{\infty} S_n$. Then $f_{n,k(n)} \to f$ uniformly on $[a,b] \setminus S'$.

We will prove the lemma by verifying that $f_{n,k(n)} \to f$ uniformly on $[a,b] \setminus S'$. Fix some $\epsilon > 0$ for verifying uniform convergence. Since $f_n \to f$ uniformly on $[a,b] \setminus S$, we can find an M_1 so that

$$n > M_1 \quad \text{implies} \quad |f(x) - f_n(x)| < \frac{\epsilon}{2} \quad \text{for } x \in [a,b] \setminus S.$$

Observe $[a,b] \setminus S \supset [a,b] \setminus S'$, so this also holds here. Now we can choose M_2 so that $\frac{1}{2^n} < \frac{\epsilon}{2}$ when $n > M_2$. Then by definition of S_n and $k(n)$, we see

$$n > M_2 \quad \text{implies} \quad |f_n(x) - f_{n,k(n)}(x)| < \frac{\epsilon}{2} \quad \text{for } x \in [a,b] \setminus S_n.$$

Again this also holds for $x \in [a,b] \setminus S'$ since this is a smaller set. Combining these two facts, we see for $n > \max\{M_1, M_2\}$ and $x \in [a,b] \setminus S'$, by the triangle inequality,

$$|f(x) - f_{n,k(n)}(x)| \leq |f(x) - f_n(x)| + |f_n(x) - f_{n,k(n)}(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have verified by definition that $f_{n,k(n)} \to f$ uniformly on $[a,b] \setminus S'$, which proves the lemma.

With the lemma we can now show that $f_{n,k(n)} \to f$ nearly uniformly. Fix some $\epsilon > 0$. Since $f_n \to f$ nearly uniformly, there is a set S with $\lambda(S) < \frac{\epsilon}{2}$ so that $f_n \to f$ uniformly on $[a,b] \setminus S$. We can also choose an $N \in \mathbb{N}$ so that

$$\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}.$$

As in the lemma, define the set

$$S' = S \cup \bigcup_{n=N}^{\infty} S_n.$$

Observe that we have arranged that $\lambda(S') < \epsilon$ by countable subadditivity. Furthermore, the lemma tells us that $f_{n,k(n)} \to f$ nearly uniformly on S', which concludes the proof.

(b) Why does (a) remain true when almost everywhere convergence replaces nearly uniform convergence?

Solution: Suppose $f_n \to f$ almost everywhere, and $f_{n,k} \to f_n$ almost everywhere. We will find a function $n \mapsto k(n)$ so that $f_{n,k(n)} \to f$ almost everywhere.

Fix n. Observe that for almost every x, $f_{n,k}(x) \to 0$. Then we can consider the set of bad points

$$B_k = \{x \in [a,b] \ : \ |f_{n,k}(x) - f_n(x)| \geq \frac{1}{2^n}\}.$$

By definition of convergence almost everywhere, we see that $\bigcap_{K\in\mathbb{N}} \bigcup_{k\geq K} B_k$ is a subset of those points which never converge to zero, and hence a zero set. The sequence of sets $\bigcup_{k\geq K} B_k$ is decreasing as K increases and is contained in $[a,b]$, so we see that $\lambda(\bigcup_{k\geq K} B_k \to 0$ as $K \to 0$. Observe $\lambda(B_K) < \lambda(\bigcup_{k\geq K} B_k$, so $\lambda(B_K) \to 0$ also. We conclude that there is a $k(n)$ so that $\lambda(B_{k(n)}) \leq \frac{1}{2^n}$. In summary, we have

$$|f_{n,k}(x) - f_n(x)| < \frac{1}{2^n} \quad \text{for } x \in [a,b] \setminus B_{k(n)} \tag{1}$$

and $\lambda(B_{k(n)}) < \frac{1}{2^n}$.

Now we may repeat the same concluding argument as before. To see that $f_{n,k(n)} \to f_n$ almost everywhere, it suffices to prove that it converges on a set of measure arbitrarily close to $\lambda([a,b])$. Fix $\epsilon > 0$. Then we can choose an N so that

$$\sum_{n=N}^{\infty} \frac{1}{2^n} < \epsilon.$$

Set $B' = \bigcup_{n=N}^{\infty} B_{k(n)}$. We have $\lambda(B') < \sum_{n=N}^{\infty} \frac{1}{2^n} < \epsilon.$. Furthermore, by equation 1, we have

$$\lim_{n\to\infty} |f_{n,k}(x) - f_n(x)| \to 0 \quad \text{for } x \in [a,b] \setminus B'.$$

Now let Z be the zero set of points x so that $f_n(x) \not\to f(x)$. Then if $x \in [a,b] \setminus Z$, we have $|f_n(x) - f(x)| \to 0$ as $n \to \infty$. We conclude that when $x \in [a,b] \setminus (B' \cup Z)$, we have

$$|f_{n,k}(x) - f(x)| \leq |f_{n,k}(x) - f_n(x)| + |f_n(x) - f(x)| < \frac{1}{2^n} + |f_n(x) - f(x)|,$$

and the right hand side tends to zero as $n \to \infty$, so the left side does as well. Observe that $\lambda\big([a,b] \setminus (B' \cup Z)\big) \geq b - a - \epsilon$, so we have shown convergence on sets arbitrarily close to full measure. Taking a countable union of such sets whose measure tends to full measure yields a set of full measure on which $f_n \to f$.