Math 70100: Functions of a Real Variable I

Homework 11, due Wednesday, November 26th.

(Email your homework in if necessary!)

1. *(Modified from Pugh, Chapter 6 # 28)* A non-negative linear combination of measurable characteristic functions is a *simple function* (or *step function*). That is, a simple function has the form
$$\phi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(c)$$
where $E_1, \ldots, E_n$ are measurable sets and $c_1, \ldots, c_n$ are non-negative constants. (The *characteristic function* of $E \subset \mathbb{R}$ is the function $\chi_E : \mathbb{R} \to \{0, 1\}$ so that $\chi_E(x) = 1$ if and only if $x \in E$.) We say that $\sum c_i \chi_{E_i}$ *expresses* ϕ. If the E_i are disjoint and the c_i are distinct and positive, then the expression for ϕ is called *canonical*.
   (a) Show that a canonical expression for a simple function exists and is unique. (*Remark: It might be useful to review part (b) to see if you want to prove more here.*)
   (b) If ϕ is a simple function with canonical representation $\sum_{i=1}^{n} c_i \chi_{E_i}$, define the "integral" $I(\phi) = \sum_i c_i \lambda(E_i)$. Show that if $\sum_{j=1}^{m} d_j \chi_{F_j}$ is a (not-necessarily canonical) expression of ϕ, then
$$I(\phi) = \sum_{j=1}^{n} d_j \lambda(F_j).$$
   (c) Infer from (b) that the map I from simple functions to $\mathbb{R}$ given by $\phi \mapsto I(\phi)$ is linear.
   (d) Given a measurable function $f : \mathbb{R} \to [0, \infty)$, show there exists a pointwise increasing sequence of simple functions $\{\phi_n\}$ whose pointwise limit is f.
   (e) Show that for any two such sequences $\{\phi_n\}$ and $\{\psi_n\}$ increasing to f as in part (d),
$$\lim_{n\to\infty} I(\phi_n) = \lim_{n\to\infty} I(\psi_n).$$
   Therefore, the definition of $I(f)$ as this limit is well-defined.
   (f) Show that the function I from the space of measurable functions $\mathbb{R} \to [0, \infty)$ to $\mathbb{R}$ given by $f \mapsto I(f)$ is linear.

2. *(Pugh, Chapter 6 # 30)* Find a sequence of measurable functions $f_n : [0, 1] \to [0, 1]$ such that $\int f_n \to 0$ as $n \to \infty$, but for no $x \in [0, 1]$ does $f_n(x)$ converge to a limit as $n \to \infty$.

3. Let $\{f_n \ : \ n \in \mathbb{N}\}$ be a sequence of measurable functions $\mathbb{R} \to [0, \infty)$. Define $g_k = \inf_{n \geq k} f_n$, i.e.,
$$g_k(x) = \inf \ \{f_n(x) \ : \ n \geq k\}.$$
Define $h = \liminf_{n\to\infty} f_n$, i.e.,
$$h(x) = \lim_{k\to\infty} g_k(x).$$
   (a) Show that g_k is measurable for all $k \in \mathbb{N}$. Explain why $\int g_k \leq \int f_n$ when $n \geq k$.

(b) Prove Fatou's lemma. Prove that h is measurable and $\int h \leq \liminf_{n\to\infty} \int f_n$. (*Hint: Use the monotone convergence theorem. Remark: Sometimes Fatou's lemma is used to prove the monotone convergence theorem, though we did not do this.*)

4. *(Pugh, Chapter 6 # 55)* A sequence of measurable functions $f_n : [a,b] \to \mathbb{R}$ converges to $f : [a,b] \to \mathbb{R}$ *nearly uniformly* if for every ϵ, there is a set $S \subset [a,b]$ with $\lambda(S) < \epsilon$ so that $f_n \to f$ uniformly on $[a,b] \setminus S$. Show that nearly uniform convergence is transitive in the following sense. Assume f_n converges to f nearly uniformly as $n \to \infty$ and that for each n there is a sequence $f_{n,k}$ which converges nearly uniformly to f_n as $k \to \infty$. (Al functions are measurable and defined on $[a,b]$.)

   (a) Show that there is a sequence $k(n) \to \infty$ such that $f_{n,k(n)}$ converges nearly uniformly to f as $n \to \infty$.

   (b) Why does (a) remain true when almost everywhere convergence replaces nearly uniform convergence?